PROBABILISTIC INEQUALITIES FOR SPECIAL CONVEX BODIES

Irini Perissinaki

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Department of Mathematics
University College London

Supervisor: Prof. Keith M. Ball

Abstract

It is proved that if the probability P is normalised Lebesgue measure on one of the ℓ_p^n balls in $\mathbf{R}^n$, then for any sequence $t_1, t_2, \ldots, t_n$ of positive numbers, the coordinate slabs $\{|x_i| \le t_i\}$ are subindependent, namely,

$$P\left(\cap_1^n \{|x_i| \le t_i\}\right) \le \prod_1^n P(\{|x_i| \le t_i\})$$

A consequence of this result is that the proportion of the volume of the unit ℓ_1^n ball which is inside the cube $[-t, t]^n$ is less than or equal to $f_n(t) = (1 - (1-t)^n)^n$.

This estimate is remarkably accurate over most of the range of values of t. A reverse inequality, demonstrating this, is the second major result of this work. A similar phenomenon occurs for all ℓ_p^n balls.

A consequence of the subindependence of the coordinate slabs of the ℓ_p^n balls, is a sort of Central Limit Theorem which is examined in the last chapter. This states that as $n \to \infty$, the average $(n-1)$-dimensional volume of the sections of the normalised ℓ_p^n ball at distance t from the origin, tends to a Gaussian. In other words, if g_θ is the density of the marginal of the ℓ_p^n-ball, in direction θ, then

$$\int_{S^{n-1}} g_\theta(t) d\sigma(\theta) \quad \longrightarrow \quad \frac{1}{\varrho\sqrt{2\pi}} \exp\left(-\frac{t^2}{2\varrho^2}\right) \qquad \text{as } n \to \infty$$

for each t, uniformly in p.

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Chapter 2 of this thesis is joint work with Prof. K.M.Ball; a modified form of it has been accepted for publication in the *Israel Journal of Mathematics*.

Contents

Introduction

Background

One of the principal problems we discuss in this piece of work, is to estimate the volume of the intersection of an Euclidean ball with a cube, in $\mathbf{R}^n$.

Even in two dimensions, the expression for the volume is rather complicated. In higher dimensions, the picture of the intersection gets quite confusing and it looks difficult to get accurate estimates just using geometric ideas. However, although probabilistic methods look more hopeful, we shall see that geometric ones do work better.

The results of our geometric approach are very surprising. Although the method is technically simple, we obtain sharp approximations over the entire range of sizes of the cube (see Theorems 2.1 and 2.2).

The original motivation for this study, was the hope that we could prove that the proportion of the volume of the unit Euclidean ball which is outside the cube $[-t, t]^n$ is of order at most $\exp\left(-\frac{nt^2}{2}\right)$. Such an estimate would imply very good lower bounds for the volume of the intersection of any sequence of central cubes. By a central cube we mean any orthogonal transformation of the cube $[-t, t]^n$. Estimates of this kind have recently been found by Gluskin.

The order of $\exp\left(-\frac{nt^2}{2}\right)$ should be set in context. A very simple geometric argument gives an order of $n \cdot \exp\left(-\frac{nt^2}{2}\right)$, as we shall explain: the gap looks small, but the advantage that would be conferred by the better bound would be considerable. The simple argument just mentioned is as follows. Let B_2^n be the Euclidean unit ball. The volume of the ball which is outside the cube $[-t, t]^n$, where $t \leq 1$, is clearly at most n times the volume of the ball which is outside of each coordinate slab $\{|x_i| \leq t\}$. Thus, the proportion is at most

$$n \cdot \frac{\int_t^1 (1-u^2)^{\frac{n-1}{2}} du}{\int_0^1 (1-u^2)^{\frac{n-1}{2}} du}$$

Now, we can use standard arguments to prove the following Theorem.

Theorem 1

$$\int_t^1 (1-u^2)^{\frac{n-1}{2}} du \quad \leq \quad \frac{1}{\sqrt{n}} \cdot \exp\left(-\frac{nt^2}{2}\right) \qquad \textit{when } nt^2 > 1 \tag{1}$$

$$\int_0^1 (1-u^2)^{\frac{n-1}{2}} du \quad \geq \quad \frac{\sqrt{\pi/8}}{\sqrt{n}} \tag{2}$$

The question of course remains. Can we do better than that? Is it possible to have an estimate of order $\exp\left(-\frac{nt^2}{2}\right)$ as we would wish? It can be seen that the question is really asking whether the overlaps between the parts of the ball outside the different slabs are large enough that the above estimate is bad. It turns out that the answer is no, to both of these questions. The factor n is absolutely necessary.

The answer is given by Theorem 2.1 below. There, it is proved that the proportion of the volume of the ball which is inside the cube $[-t,t]^n$ is less than, or equal to the n-th power of the proportion of its volume that is inside one coordinate slab of width t. That is,

$$\frac{Vol_n(B_2^n \cap [-t,t]^n)}{Vol_n(B_2^n)} \leq \left[\frac{Vol_n(B_2^n \cap \{|x_1| \leq t\})}{Vol_n(B_2^n)}\right]^n$$

or equivalently,

$$\frac{Vol_n(B_2^n \cap [-t,t]^n)}{Vol_n(B_2^n)} \leq \left[\frac{\int_0^t (1-u^2)^{\frac{n-1}{2}} du}{\int_0^1 (1-u^2)^{\frac{n-1}{2}} du}\right]^n$$

This gives a lower bound for the proportion of the volume of the ball which is outside the cube:

$$\frac{Vol_n(B_2^n \backslash [-t,t]^n)}{Vol_n(B_2^n)} \geq 1 - \left[\frac{\int_0^t (1-u^2)^{\frac{n-1}{2}} du}{\int_0^1 (1-u^2)^{\frac{n-1}{2}} du}\right]^n$$

It is not hard to check that when $nt^2 \approx 2\log n$, the above lower bound, is roughly a constant. So, for this value of t, it is of order $n\exp\left(-\frac{nt^2}{2}\right)$. To see it, we can use the upper bound $1-\frac{1}{4}(1-t^2)^{n/2}$ for $\int_0^t (1-u^2)^{\frac{n-1}{2}} du / \int_0^1 (1-u^2)^{\frac{n-1}{2}} du$ which is discussed in Lemma A.1. When $t^2 \approx 2\frac{\log n}{n}$ this implies that

$$\frac{\int_0^t (1-u^2)^{\frac{n-1}{2}} du}{\int_0^1 (1-u^2)^{\frac{n-1}{2}} du} \leq 1 - \frac{1}{4n}$$

and thus,

$$\begin{aligned}\frac{Vol_n(B_2^n \backslash [-t,t]^n)}{Vol_n(B_2^n)} &\geq 1 - \left(1 - \frac{1}{4n}\right)^n \\ &\geq 1 - \mathrm{e}^{-1/4}\end{aligned}$$

The main Theorems

Although our original hope of better estimates for Euclidean balls turned out to be false, our research in this direction, led us to obtain information about the proportion of a general ℓ_p^n ball, inside a cube, which was far more accurate than we could possibly have expected. This information in turn, very quickly provides a "Central Limit Theorem" described below.

Recall that the unit ℓ_p^n ball which is denoted by B_p^n is the set $\{x = (x_1, \ldots, x_n) \in \mathbf{R}^n : \sum_{i=1}^n |x_i|^p \leq 1\}$. Several works have appeared in the past, estimating the volumes of intersections of ℓ_p balls. The most noteworthy, is that of Schechtman and Zinn, [3]. There, they deal with the more general problem of estimating the proportion of the volume left in the ℓ_p^n ball after removing a t-multiple of the ℓ_q^n ball, when $p < q$. They prove that this proportion is of order $\exp(-cnt^p)$. Taking limits as $q \longrightarrow \infty$, they also mention some results about the proportion of the volume of the ℓ_p^n ball which is outside the cube $[-t, t]^n$. Their results in this particular case, which is the one that interests us, are summed up in the following two statements. (Their first result in this particular range is rather weak, as it is a particular case of the easy geometric argument mentioned before, but in the more general setting they consider, there is no such obvious approach.)

$$\begin{aligned} &\text{If} \quad t \geq \tau \left(\frac{\log n}{n}\right)^{1/p} &&, \text{then} \quad P(\{||x||_\infty \geq t\}) \leq \exp\left(-\gamma n t^p / p\right) \\ &\text{and if} \quad \frac{2}{n^{1/p}} \leq t \leq \frac{1}{2} &&, \text{then} \quad P(\{||x||_\infty \geq t\}) \geq \exp\left(-\Gamma n t^p / p\right) \end{aligned}$$

where γ, Γ and τ are universal constants.

In the case $q = \infty$ considered here, our results are much stronger.

Our first main Theorem is the subindependence of coordinate slabs, stated below as Theorem 2.1.

Theorem 2.1 (Subindependence of coordinate slabs) *If the probability P is normalised Lebesgue measure on one of the ℓ_p^n balls in $\mathbf{R}^n$, then for any sequence $t_1, \ldots, t_n$ of positive numbers,*

$$P\left(\cap_1^n \{|x_i| \leq t_i\}\right) \leq \prod_1^n P(\{|x_i| \leq t_i\}).$$

Taking $t_1 = \ldots = t_n = t$ we get an upper bound for the proportion of the volume of the unit ℓ_p^n ball which is inside the cube $[-t, t]^n$. Our second main Theorem, Theorem 3.1, shows that this bound is a very good approximation to this proportion. For simplicity, we shall illustrate this only in the case $p = 1$. In this case, the upper bound is given in the Corollary bellow:

Corollary 2.1.1 *If $F_n(t)$ is the proportion of the volume of the ℓ_1^n ball inside the cube $[-t, t]^n$ then*

$$F_n(t) \leq f_n(t) = (1 - (1 - t)^n)^n$$

This upper bound is extremely precise as long as $F_n(t)$ is not too small. The easiest way to state this is to write it as an estimate for the volume outside the cube, namely for $1 - F_n(t)$.

Theorem 2.2 (An estimate in the reverse direction) *With $F_n(t)$ as above,*

$$\frac{1-F_n(t)}{1-f_n(t)} = 1 + O\left(\frac{(\log n)^3}{n}\right)$$

as $n \to \infty$ uniformly in t.

Theorem 2.2 enables us to describe the threshold behaviour of $F_n(t)$ much more precisely than Schechtman and Zinn. For example, if $t = \frac{\log n - \log c}{n}$ then the information we get from Theorem 2.2 is that $F_n(t)$ should be something like $f_n(t)$, which in turn is something like

$$(1 - \exp(-\log n + \log c))^n = \left(1 - \frac{c}{n}\right)^n \simeq \exp(-c).$$

In the last Chapter, we prove the third main Theorem of this work, Theorem 6.1. This is a sort of Central Limit Theorem, which is almost entirely an application of the subindependence of coordinate slabs. This states that as $n \to \infty$, the average $(n-1)$-dimensional volume of the sections of the normalised ℓ_p^n ball at distance t from the origin, tends to a Gaussian:

Theorem 6.1 *If g_θ is the density of the marginal in direction θ, of the ℓ_p^n-ball, then*

$$\int_{S^{n-1}} g_\theta(t) d\sigma(\theta) \quad \longrightarrow \quad \frac{1}{\varrho\sqrt{2\pi}} \exp\left(-\frac{t^2}{2\varrho^2}\right) \qquad \textit{as } n \to \infty$$

Organisation of the thesis

In Chapter 1, we start with some "easy" estimates for the proportion of the volume of the ℓ_1^n ball inside a cube. These already answer (in the negative) the question discussed in the first part of this introduction.

With "F" as above, the result proved there, describes the asymptotic behaviour of $F_n\left(1-\left(\frac{x}{n}\right)^{\frac{1}{n}}\right)$. As $n\to\infty$,

$$F_n\left(1-\left(\frac{x}{n}\right)^{\frac{1}{n}}\right)\longrightarrow \mathrm{e}^{-x}$$

It is easy to see that when $1-\left(\frac{x}{n}\right)^{1/n}\approx\frac{\log n}{n}$, the above relation implies that

$$F_n\left(\frac{\log n}{n}\right)\approx \mathrm{e}^{-1}$$

which rules out the possibility of removing the "n factor".

In Chapter 2 we give the proofs of the first two main Theorems mentioned above, for the most simple case $p=1$. In the next Chapter are given the proofs of their general cases. Where the generalisation is quite similar, we just give a brief sketch of the proof. This simplifying strategy, is the reason we illustrate the most easy case $p=1$ separately.

In Chapter 4 we also prove a result similar to Theorem 2.1, the subindependence of the complements of coordinate slabs:

$$P\left(\cap_1^n\{|x_i|\geq t_i\}\right)\leq\prod_1^n P(\{|x_i|\geq t_i\}).$$

A counterexample is given in Chapter 5 showing that the subindependence of coordinate slabs is a property depending heavily upon the ℓ_p^n balls and not applicable for highly symmetric convex bodies in general. This check was prompted by the realisation that the Theorem of Whitney-Loomis in the case of the ℓ_p^n balls, is a limiting case of the subindependence of coordinate slabs.

Finally, in Chapter 6 we prove Theorem 6.1, which as we mentioned above, is a sort of Central Limit Theorem.

Chapter 1

An easy estimate

Theorem 2.1 states that $F_n(t)$ is dominated by the function $(1-(1-t)^n)^n$, or if we put $x = n(1-t)^n$, $(0 < x < n)$,

$$F_n\left(1-\left(\frac{x}{n}\right)^{\frac{1}{n}}\right) \le \left(1-\frac{x}{n}\right)^n$$

In Theorem 2.2 it is proved that this is a very precise inequality.

Here, we are going to prove, by rather easy means, that the two functions are asymptotically the same; namely that as $n \longrightarrow \infty$,

$$F_n\left(1-\left(\frac{x}{n}\right)^{\frac{1}{n}}\right) \longrightarrow \mathrm{e}^{-x}$$

Our argument uses the exact formula for $F_n\left(1-\left(\frac{x}{n}\right)^{\frac{1}{n}}\right)$ proved in Theorem 1.2. The idea is to notice that each term of this series, converges to the corresponding term of the series of e^{-x}. Then we apply the Dominated Convergence Theorem, to get the limit for the series.

1.1 The asymptotic behaviour

Theorem 1.2, proved below shows that the proportion of the ℓ_1^n ball inside the cube $[-t,t]^n$, is

$$F_n(t) = \sum_{j=0}^{n} (-1)^j \binom{n}{j} (1-jt)_+^n$$

where as usual

$$x_+ = \begin{cases} x & \text{when} \quad x \geq 0 \\ 0 & \text{when} \quad x < 0. \end{cases}$$

If we put $\binom{n}{j} = 0$ when $n < j$, this can be written

$$F_n(t) = \sum_{j=0}^{\infty} (-1)^j \binom{n}{j} (1-jt)_+^n$$

We then want to prove the following:

Theorem 1.1 *For all positive numbers x*

$$\sum_{j=0}^{\infty} (-1)^j \binom{n}{j} \left(1 - j\left(1 - \left(\frac{x}{n}\right)^{\frac{1}{n}}\right)\right)_+^n \quad \longrightarrow \quad \sum_{j=0}^{\infty} (-1)^j \frac{x^j}{j!} \qquad \textit{as} \quad n \longrightarrow \infty$$

Proof: We shall prove that for x and j fixed we have:

$$\binom{n}{j} \left(1 - j\left(1 - \left(\frac{x}{n}\right)^{\frac{1}{n}}\right)\right)_+^n \quad \leq \quad \frac{x^j}{j!} \tag{1.1}$$

and

$$\binom{n}{j} \left(1 - j\left(1 - \left(\frac{x}{n}\right)^{\frac{1}{n}}\right)\right)_+^n \quad \longrightarrow \quad \frac{x^j}{j!} \qquad \text{as} \quad n \longrightarrow \infty \tag{1.2}$$

Once (1.1) and (1.2) are proved, we need only apply The Dominated Convergence Theorem.

We shall first prove (1.1). Since $\binom{n}{j} \leq \frac{n^j}{j!}$, it is enough to prove that:

$$n^j \left(1 - j\left(1 - \left(\frac{x}{n}\right)^{\frac{1}{n}}\right)\right)^n \quad \leq \quad x^j \tag{1.3}$$

whenever j is less than or equal to the integer part of $\left(1 - \left(\frac{x}{n}\right)^{\frac{1}{n}}\right)^{-1}$. Using the fact that

$$1 - js \leq (1-s)^j \qquad \text{when} \quad 0 \leq s \leq 1\,, \quad j \geq 1$$

for $s = 1 - \left(\frac{x}{n}\right)^{\frac{1}{n}}$, we get:

$$1 - j\left(1 - \left(\frac{x}{n}\right)^{\frac{1}{n}}\right) \ \leq \ \left(\frac{x}{n}\right)^{\frac{j}{n}}$$

or equivalently

$$n^j \left(1 - j\left(1 - \left(\frac{x}{n}\right)^{\frac{1}{n}}\right)\right)^n \ \leq \ x^j$$

which is what we want.

For the proof of (1.2), we again notice that since $\frac{j!}{n^j}\binom{n}{j} \to 1$ it is enough to prove that

$$n^j \left(1 - j\left(1 - \left(\frac{x}{n}\right)^{\frac{1}{n}}\right)\right)^n \quad \longrightarrow \quad x^j \qquad \text{as} \quad n \longrightarrow \infty \tag{1.4}$$

Equation (1.1) already gives us an upper bound. To get the lower bound we notice first that since $\log t \leq t - 1$, we have

$$\begin{aligned}
n^j \left(1 - j\left(1 - \left(\frac{x}{n}\right)^{\frac{1}{n}}\right)\right)^n &\geq n^j \left(1 - \frac{j}{n}(\log n - \log x)\right)^n \\
&= n^j \left(1 + \frac{j\log x}{n} - \frac{j\log n}{n}\right)^n \\
&\geq n^j \left(1 + \frac{j\log x}{n}\right)^n \left(1 - \frac{j\log n}{n}\right)^n \\
&\geq \left(1 + \frac{j\log x}{n}\right)^n \left(1 - \left(\frac{j\log n}{n}\right)^2\right)^n
\end{aligned}$$

which converges to x^j as $n \longrightarrow \infty$.

1.2 A formula for $F_n(t)$

In this section we prove that the proportion of the volume of the unit ℓ_1^n ball inside the cube $[-t,t]^n$, which we denote, $F_n(t)$, equals

$$\sum_{j=0}^{n}(-1)^j\binom{n}{j}(1-jt)_+^n$$

The proof uses probabilistic arguments, but as will become clear later, geometric ones could just as well be used.

We first prove a Lemma which simply states that we can write down a formula for the n-fold convolution $g_n = \underbrace{g * g * \cdots * g}_{n \text{ times}}$ of the uniform distribution function in [0,1], g.

Lemma 1.1 *If $g_1(s) = \mathbf{1}_{[0,1]}(s)$ and $g_n(s) = \int_{s-1}^{s} g_{n-1}(u)du$, then, for $n \geq 2$ and $s \geq 0$,*

$$g_n(s) = \frac{1}{(n-1)!}\sum_{j=0}^{n}(-1)^j\binom{n}{j}(s-j)_+^{n-1}$$

Proof: We prove this by induction:

Observe that for any $j \geq 0$

$$\int_0^t (u-j)_+^{n-1}\,du = \frac{1}{n}(t-j)_+^n.$$

Assuming the formula for g_n and using the fact that

$$g_{n+1}(s) = \int_0^s g_n(u)\,du - \int_0^{s-1} g_n(u)\,du$$

we get

$$
\begin{aligned}
g_{n+1}(s) &= \frac{1}{n!}\left(\sum_{j=0}^{n}(-1)^j\binom{n}{j}(s-j)_+^n - \sum_{j=0}^{n}(-1)^j\binom{n}{j}(s-1-j)_+^n\right)\\
&= \frac{1}{n!}\left(\sum_{j=0}^{n}(-1)^j\binom{n}{j}(s-j)_+^n + \sum_{j=1}^{n+1}(-1)^j\binom{n}{j-1}(s-j)_+^n\right)\\
&= \frac{1}{n!}\left(\sum_{j=0}^{n+1}(-1)^j\binom{n+1}{j}(s-j)_+^n\right)
\end{aligned}
$$

We next prove the formula for F_n:

Theorem 1.2

$$F_n(t) = \sum_{j=0}^{n}(-1)^j\binom{n}{j}(1-jt)_+^n \qquad \textit{for all } t \geq 0$$

Proof: Recall that by $F_n(t)$, we denote the proportion of the volume of the unit ℓ_1^n ball which is inside the cube $[-t,t]^n$. For convenience, write $F_n^*(s)$ for the proportion of the volume of the ℓ_1^n ball of radius s which is inside the cube $[-1,1]^n$. Obviously F_n and F_n^* are related. Indeed

$$F_n(t) = F_n^*(1/t)$$

Thus, our aim is to prove that

$$F_n^*(s) = \sum_{j=0}^{n}(-1)^j\binom{n}{j}\left(1-\frac{j}{s}\right)_+^n \tag{1.5}$$

This being trivial for $n=1$, we shall prove it for $n \geq 2$.

Write Q_n^+ for the part of the cube which is in the positive orthant, that is, $Q_n^+ = [0,1]^n$. Clearly

$$F_n^*(s) = \frac{Vol_n(B_1^n(s)\cap Q_n^+)}{Vol_n(B_1^n(s)\cap \mathbf{R}_+^n)} \tag{1.6}$$

Now let $X_1, \ldots, X_n$ be independent, identically distributed random variables, each uniformly distributed on $[0,1]$. Then the vector $(X_1, \ldots, X_n)$ induces Lebesgue measure on Q_n^+

It is easy to see that if $\overrightarrow{e}$ is the unit vector $(1/\sqrt{n}, \ldots, 1/\sqrt{n})$, and H the hyperplane $\langle \overrightarrow{e} \rangle^{\perp}$, then,

$$\begin{aligned} Vol_{n-1}\left(Q_n^+ \cap (H + s\overrightarrow{e})\right) &= \text{density of } \left(\frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i\right) \text{ at point } s \\ &= \sqrt{n}\left(\text{density of } \sum_{i=1}^{n} X_i \text{ at point } \sqrt{n}s\right) \end{aligned}$$

So, for the volume $Vol_n(B_1^n(s) \cap Q_n^+)$ we have:

$$\begin{aligned} Vol_n(B_1^n(s) \cap Q_n^+) &= \int_0^{s/\sqrt{n}} Vol_{n-1}\left(Q_n^+ \cap (H + u\overrightarrow{e})\right) du \\ &= \int_0^{s/\sqrt{n}} \sqrt{n}\left(\text{density of } \sum_{i=1}^{n} X_i \text{ at point } \sqrt{n}u\right) du \\ &= \int_0^{s} \left(\text{density of } \sum_{i=1}^{n} X_i \text{ at point } u\right) du \qquad (1.7) \end{aligned}$$

Thus, all we need to do, is to calculate the density of $\sum_{i=1}^{n} X_i$. By independence the density of the sum is the convolution of the densities of the X_i's, $\underbrace{g * g * \cdots * g}_{n \text{ times}}$. If we write g_n for $\underbrace{g * g * \cdots * g}_{n \text{ times}}$, we obtain

$$g_n(s) = \int_{s-1}^{s} g_{n-1}(u)du$$

and of course, $g_1 = \mathbf{1}_{[0,1]}$. By Lemma 1.1, we have that for $u \geq 0$,

$$g_n(u) = \frac{1}{(n-1)!} \sum_{j=0}^{n} (-1)^j \binom{n}{j} (u-j)_+^{n-1} \qquad (1.8)$$

So, by integrating (1.8), we get from (1.7) that:

$$Vol_n(B_1^n(s) \cap Q_n^+) = \frac{1}{n!} \sum_{j=0}^{n} (-1)^j \binom{n}{j} (s-j)_+^n$$

Hence, dividing by the volume of the whole corner,

$$F_n^*(s) = \frac{Vol_n(B_1^n(s) \cap Q_n^+)}{Vol_n(B_1^n(s) \cap \mathbf{R}_+^n)} = \sum_{j=1}^{n} (-1)^j \binom{n}{j} \left(1 - \frac{j}{s}\right)_+^n$$

Remarks

1. We notice that

$$F_n^*(s) = ns^{-n} \int_{s-1}^{s} u^{n-1} F_{n-1}^*(u) du \tag{1.9}$$

Indeed, by (1.6) and (1.7),

$$\int_0^s g_{n-1}(u) du = \frac{s^{n-1}}{(n-1)!} \cdot F_{n-1}^*(s)$$

thus,

$$\int_{u-1}^{u} g_{n-1}(x) dx = \frac{u^{n-1}}{(n-1)!} \cdot F_{n-1}^*(u) - \frac{(u-1)^{n-1}}{(n-1)!} \cdot F_{n-1}^*(u-1)$$

or

$$g_n(u) = \frac{u^{n-1}}{(n-1)!} \cdot F_{n-1}^*(u) - \frac{(u-1)^{n-1}}{(n-1)!} \cdot F_{n-1}^*(u-1)$$

and so,

$$\begin{aligned}
F_n^*(s) &= \frac{Vol_n(B_1^n(s) \cap Q_n^+)}{Vol_n(B_1^n(s) \cap \mathbf{R}_+^n)} \\
&= \frac{n!}{s^n} \int_0^s g_n(u) du \\
&= \frac{n!}{s^n} \int_0^s \left(\frac{u^{n-1}}{(n-1)!} \cdot F_{n-1}^*(u) - \frac{(u-1)^{n-1}}{(n-1)!} \cdot F_{n-1}^*(u-1) \right) \\
&= \frac{n}{s^n} \left(\int_0^s u^{n-1} F_{n-1}^*(u) du - \int_{-1}^{s-1} u^{n-1} F_{n-1}^*(u) du \right) \\
&= ns^{-n} \int_{s-1}^{s} u^{n-1} F_{n-1}^*(u) du
\end{aligned}$$

Now, if we put $t = 1/s$ in (1.9) we get:

$$F_n(t) = nt^n \int_{1/t-1}^{1/t} u^{n-1} F_{n-1}(1/u) du$$

If we substitute $u = \frac{1-v}{t} \Leftrightarrow v = 1 - tu$ the above relation becomes:

$$F_n(t) = n \int_0^t (1-v)^{n-1} F_{n-1}\left(\frac{t}{1-v}\right) dv. \qquad (1.10)$$

This relation will reappear later as (2.1).

2. We could use (1.9), as a recurrence relation for F_n^*, to find its precise formula, in a rather more direct way than first finding g_n. Since (1.10) was proved in a geometric way and since it is equivalent to (1.9), this means that one could use a geometric rather than a probabilistic argument to find the formula for F_n.

3. The formula for F_n can be also obtained by just integrating the differential equation (2.2) (see Chapter 2).

Chapter 2

Coordinate slabs of the ℓ_1^n ball

In this Chapter we give a detailed proof for the simplest case of Theorem 2.1 (the subindependence of coordinate slabs) and Theorem 3.1 (an estimate in the reverse direction); namely the case $p = 1$.

Theorem 2.1 (Subindependence of coordinate slabs) *If the probability P is normalised Lebesgue measure on one of the ℓ_p^n balls in $\mathbf{R}^n$, then for any sequence $t_1, \ldots, t_n$ of positive numbers,*

$$P\left(\cap_1^n \{|x_i| \leq t_i\}\right) \leq \prod_1^n P(\{|x_i| \leq t_i\}).$$

The particular case $p = 1,\ t_1 = \ldots = t_n$ of Theorem 2.1 gives an upper bound for the proportion of the volume of the ℓ_1^n ball which is inside the cube $[-t,t]^n$. Since the proportion of the volume of the ℓ_1^n ball which is inside a coordinate slab of width $2t$ is $1-(1-t)^n$ when $t \leq 1$, the result in this case is given by the following Corollary.

Corollary 2.1.1 *If $F_n(t)$ is the proportion of the volume of the ℓ_1^n ball inside the cube $[-t,t]^n$ then*

$$F_n(t) \leq f_n(t) = (1-(1-t)^n)^n$$

Although $F_n(t)$ is the function $\sum_0^n (-1)^j \binom{n}{j}(1-jt)_+^n$, (see Chapter 1), which is a spline with many knots, we prove in Theorem 2.2 (the particular case of Theorem 3.1) that the polynomial $f_n(t) = (1-(1-t)^n)^n$ is an astonishingly good approximation to $F_n(t)$, at least when $F_n(t)$ is not too small. The easiest way to state this is to write it as an estimate for the volume outside the cube, namely for $1 - F_n(t)$.

Theorem 2.2 (An estimate in the reverse direction) *With $F_n(t)$ as above,*

$$\frac{1-F_n(t)}{1-f_n(t)} = 1 + O\left(\frac{(\log n)^3}{n}\right)$$

as $n \to \infty$, uniformly in t.

2.1 Method

In this section we will briefly explain the crucial points of the proofs of these two Theorems for the simplest case when $p = 1$ and $t_1 = \cdots = t_n = t$.

The proof of Theorem 2.1, (the upper bound for F_n) depends on a very convenient interaction between two different equations expressing F_n and its derivative in terms of F_{n-1}. Each of these equations is proved using a simple geometric argument: they can readily be combined to give a differential inequality for F_n which integrates up to the stated result.

These equations are:

$$F_n(y) = n \int_0^y (1-u)^{n-1} F_{n-1}\left(\frac{y}{1-u}\right) du$$

$$\frac{d}{dy} F_n(y) = n^2 (1-y)^{n-1} F_{n-1}\left(\frac{y}{1-y}\right)$$

The proof of Theorem 2.2, (a lower bound for F_n) is technically more complicated although it is much less delicate. The crucial point is to show that at its maximum, the function $\frac{1-F_n}{1-f_n}$ is dominated by the value of a related function, which in turn can be shown to be small by means of the (rather precise) upper bound, already proved.

In fact, this related function, say $G_n(t)$, is not as small as we would like it to be in the whole interval (0,1), but it behaves nicely in a smaller interval $[t_n, 1/2]$, for some value of t_n which is roughly like $\frac{\log n - \log\log n}{n}$. It is in this range that $\frac{1-F_n}{1-f_n}$ actually attains its maximum. However, for technical reasons, it is simpler to show directly that $\frac{1-F_n}{1-f_n}$ is small outside this interval.

2.2 The upper bound.

In this section we shall give a detailed proof of Theorem 2.1 in the case $p = 1$.

- *Proof of Theorem* 2.1 *for the case* $p = 1$, $t_1 = \ldots = t_n = t$:

Except in the trivial case $t \geq 1$ the problem is to show that the proportion of the volume of the unit ℓ_1^n ball which is inside the cube $Q_n(t) = [-t, t]^n$ is bounded from above by the function $f_n(t) = (1 - (1 - t)^n)^n$. This proportion will be denoted by $F_n(t)$.

The proof uses the following two equations:

$$F_n(y) = n \int_0^y (1-u)^{n-1} F_{n-1}\left(\frac{y}{1-u}\right) du \tag{2.1}$$

$$\frac{d}{dy} F_n(y) = n^2 (1-y)^{n-1} F_{n-1}\left(\frac{y}{1-y}\right) \tag{2.2}$$

Assuming these, we proceed as follows. Since F_{n-1} is an increasing function, $F_{n-1}\left(\frac{y}{1-u}\right)$ is increasing in u. So from (2.1) we get:

$$F_n(t) \leq n F_{n-1}\left(\frac{y}{1-y}\right) \int_0^y (1-u)^{n-1} du$$

For convenience, we shall abbreviate the integral $\int_0^y (1-u)^{n-1} du$ by $Y_n(y)$. Then (2.2) and the inequality can be written,

$$F_n(y) \leq n F_{n-1}\left(\frac{y}{1-y}\right) Y_n(y) \tag{2.3}$$

$$\frac{d}{dy} F_n(y) = n^2 F_{n-1}\left(\frac{y}{1-y}\right) \frac{d}{dy} Y_n(y) \tag{2.4}$$

If we eliminate the factor $n F_{n-1}\left(\frac{y}{1-y}\right)$ we get:

$$\frac{\frac{d}{dy} F_n(y)}{F_n(y)} \geq n \frac{\frac{d}{dy} Y_n(y)}{Y_n(y)} \tag{2.5}$$

and, by integrating from t to 1 we get the desired result:

$$F_n(t) \leq \left(\frac{Y_n(t)}{Y_n(1)}\right)^n = (1 - (1-t)^n)^n$$

It remains to prove the relations (2.1) and (2.2).

For the first one,

$$
\begin{aligned}
F_n(y) &= \frac{Vol_n(Q_n(y) \cap B_1^n)}{Vol_n(B_1^n)} \\
&= \frac{n!}{2^n} 2 \int_0^y Vol_{n-1}(Q_n(y) \cap B_1^n \cap \{x_1 = u\}) du \\
&= \frac{n!}{2^{n-1}} \int_0^y Vol_{n-1}(Q_{n-1}(y) \cap B_1^{n-1}(1-u)) du \\
&= n \int_0^y (1-u)^{n-1} \frac{Vol_{n-1}(Q_{n-1}(y) \cap B_1^{n-1}(1-u))}{Vol_{n-1}(B_1^{n-1}(1-u))} du \\
&= n \int_0^y (1-u)^{n-1} F_{n-1}\left(\frac{y}{1-u}\right) du
\end{aligned}
$$

For the second one, put $H_n(y) = Vol_n(Q_n(y) \cap B_1^n)$. Since $F_n(y) = \frac{H_n(y)}{Vol_n(B_n^1)} = \frac{n!}{2^n} H_n(y)$, to find $\frac{d}{dy} F_n(y)$ it suffices to find $\frac{d}{dy} H_n(y)$, which is;

$$
\begin{aligned}
\frac{d}{dy} H_n(y) &= \lim_{h \to 0} \frac{H_n(y+h) - H_n(y)}{h} \\
&= 2n Vol_{n-1}(Q_{n-1}(y) \cap B_1^{n-1}(1-y))
\end{aligned}
$$

and thus,

$$
\begin{aligned}
\frac{d}{dy} F_n(y) &= n^2 \frac{Vol_{n-1}(Q_{n-1}(y) \cap B_1^{n-1}(1-y))}{Vol_{n-1}(B_1^{n-1})} \\
&= n^2 (1-y)^{n-1} \frac{Vol_{n-1}(Q_{n-1}(y) \cap B_1^{n-1}(1-y))}{Vol_{n-1}(B_1^{n-1}(1-y))} \\
&= n^2 (1-y)^{n-1} F_{n-1}\left(\frac{y}{1-y}\right)
\end{aligned}
$$

- *Proof of Theorem* 2.1 *for the case* $p = 1$:

For convenience, let $F_n(t_1, \ldots, t_n)$ denote the proportion of the volume of the unit ℓ_1^n ball which is inside the cuboid $Q_n(t_1, \ldots, t_n) = [-t_1, t_1] \times$

$\ldots \times [-t_n, t_n]$. The Theorem states that

$$F_n(t_1, \ldots, t_n) \leq \frac{Y_n(t_1)}{Y_n(1)} \cdots \frac{Y_n(t_n)}{Y_n(1)}$$

where $Y_n(t)$ is the integral $\int_0^{min\{1,t\}} (1-u)^{n-1} du$

Of course, if one of the t_i's is zero, then $F_k(t_1, \ldots, t_k) = 0$ and the inequality is trivial. It is also trivial when all the t_i's are greater than 1.

If neither of these trivial cases applies, we prove that as long as for some i the t_i is less than 1, the value of the function F_n at point $(t_1, \ldots, t_n)$ is dominated by an appropriate multiple of the value of F_n, at the point with the ith coordinate replaced by 1 and the rest remaining the same, i.e.

$$F_n(t_1, \ldots, t_n) \leq \frac{Y_n(t_i)}{Y_n(1)} F_n(t_1 \ldots, t_{i-1}, 1, t_{i+1} \ldots, t_n) \tag{2.6}$$

So, if we suppose, without loss of generality, that $0 < t_i < 1$ for $i = 1 \ldots k$, $(1 < k \leq n)$ and $t_i \geq 1$ for $i = k+1, \ldots, n$, then we will have in turn the following inequalities:

$$\begin{aligned} F_n(t_1, \ldots, t_n) &\leq \frac{Y_n(t_1)}{Y_n(1)} F_n(1, t_2 \ldots, t_n) \\ &\leq \frac{Y_n(t_1)}{Y_n(1)} \frac{Y_n(t_2)}{Y_n(1)} F_n(1, 1, t_3 \ldots, t_n) \\ & \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &\leq \frac{Y_n(t_1)}{Y_n(1)} \cdots \frac{Y_n(t_k)}{Y_n(1)} F_n(1, \ldots, 1, t_{k+1}, \ldots, t_n) \end{aligned}$$

Since $F_n(1,\ldots,1,t_{k+1}\ldots,t_n) = 1$ the proof is complete.

Thus, the crucial point is to prove (2.6). Without loss of generality, we will prove this for $i = 1$, namely the relation:

$$F_n(t_1,\ldots,t_n) \leq \frac{Y_n(t_1)}{Y_n(1)} F_n(1,t_2\ldots,t_n) \tag{2.7}$$

when $0 < t_1 < 1$.

To do this, we again combine two equations. The first one relates F_n and F_{n-1}, and the second one relates F_{n-1} and the partial derivative of F_n with respect to the first coordinate. These are:

$F_n(y,t_2,\ldots,t_n) =$

$$\begin{aligned} &= n\int_0^y (1-u)^{n-1} F_{n-1}\left(\frac{t_2}{1-u},\ldots,\frac{t_n}{1-u}\right) du \qquad (2.8)\\ &\leq nF_{n-1}\left(\frac{t_2}{1-y},\ldots,\frac{t_n}{1-y}\right)\int_0^y (1-u)^{n-1}du \\ &= nF_{n-1}\left(\frac{t_2}{1-y},\ldots,\frac{t_n}{1-y}\right) Y_n(y) \end{aligned}$$

and

$$\begin{aligned} \frac{d}{dy}F_n(y,t_2,\ldots,t_n) &= n(1-y)^{n-1}F_{n-1}\left(\frac{t_2}{1-y},\ldots,\frac{t_n}{1-y}\right) \qquad (2.9)\\ &= F_{n-1}\left(\frac{t_2}{1-y},\ldots,\frac{t_n}{1-y}\right)\frac{d}{dy}Y_n(y) \end{aligned}$$

Eliminating $F_{n-1}\left(\frac{t_2}{1-y},\ldots,\frac{t_n}{1-y}\right)$, we get

$$\frac{\frac{d}{dy}F_n(y,t_2,\ldots,t_n)}{F_n(y,t_2,\ldots,t_n)} \geq \frac{\frac{d}{dy}Y_n(y)}{Y_n(y)} \tag{2.10}$$

which integrates to (2.7).

The proofs of (2.8) and (2.9) are similar to the proofs of (2.1) and (2.2).

Remarks

1. (2.5) and (2.10) actually state that the functions $\frac{F_n(y)}{(Y_n(y))^n}$ and $\frac{F_n(y,t_2,\ldots,t_n)}{Y_n(y)}$, are increasing in y.

   A consequence of this, is that the function $\frac{F_n(t_1,\ldots,t_n)}{Y_n(t_1)\ldots Y_n(t_n)}$ is increasing in each coordinate.

2. If $0 < t_i < 1$ for $i = 1\ldots k$, $(1 < k \leq n)$ and $t_i \geq 1$ for $i = k+1,\ldots,n$, then Theorem 2.1 states that

$$F_n(t_1,\ldots,t_n) \leq (1-(1-t_1)^n)\ldots(1-(1-t_k)^n)$$

2.3 The lower bound

Using the notation introduced in the previous section, we shall prove that the function $f_n(t)$ is not only an upper bound (see Theorem 2.1), but it is also a very good approximation to $F_n(t)$, within the interesting range of t. More precisely, we prove that the function $\frac{1-F_n(t)}{1-f_n(t)}$ converges to 1 uniformly in t, as stated in the next Theorem;

Theorem 2.1 (An estimate in the reverse direction)

$$\frac{1-F_n(t)}{1-f_n(t)} = 1 + O\left(\frac{(\log n)^3}{n}\right)$$

uniformly in t.

We focus our attention on the point t_{max}, where $\frac{1-F_n(t)}{1-f_n(t)}$ attains its maximum value. In the first Lemma below, we find a function $G_n(t)$ which dominates $\frac{1-F_n(t)}{1-f_n(t)}$ at t_{max}. This related function, is proved to be small in a particular range, where t_{max} actually occurs. Outside this range $\frac{1-F_n(t)}{1-f_n(t)}$ is small for very simple reasons. To avoid technical difficulties, we don't actually prove that t_{max} is in this particular range.

Lemma 2.1 *At its maximum point, the function* $\frac{1-F_n(t)}{1-f_n(t)}$ *is dominated by the value of the function* $G_n(t) = \begin{cases} \left[\frac{1-\left(1-\frac{t}{1-t}\right)^{n-1}}{1-(1-t)^n}\right]^{n-1} & , \quad 0 < t \leq 1/2 \\ [1-(1-t)^n]^{-(n-1)} & , \quad 1/2 < t < 1 \end{cases}$

Proof of Lemma 2.1: Before embarking upon the proof it is perhaps worth mentioning that it depends critically upon Theorem 2.1 (the upper bound for F_n) already proved.

It is easy to check that $\frac{1-F_n(t)}{1-f_n(t)} \to 1$ as $t \to 0$ or $t \to 1$. So $\frac{1-F_n}{1-f_n}$ attains its maximum in (0,1).

So

$$\left[\frac{d}{dt}\log\left(\frac{1-F_n(t)}{1-f_n(t)}\right)\right]_{t_{max}} = 0$$

i.e.

$$\frac{1-F_n(t_{max})}{1-f_n(t_{max})} = \frac{\frac{d}{dt}F_n(t_{max})}{\frac{d}{dt}f_n(t_{max})}$$

But $\frac{d}{dt}F_n(t)$ has already been calculated in (2.2). Substituting this in the above relation, as well as $\frac{d}{dt}f_n(t_{max})$ we get that

$$\frac{1-F_n(t_{max})}{1-f_n(t_{max})} = \frac{F_{n-1}\left(\frac{t_{max}}{1-t_{max}}\right)}{(1-(1-t_{max})^n)^{n-1}}$$

Of course, $F_{n-1}\left(\frac{t_{max}}{1-t_{max}}\right) = 1$ if $1/2 < t_{max} < 1$.

To prove the required inequality for $0 < t_{max} \leq 1/2$, it is sufficient to apply Theorem 2.1 in order to dominate $F_{n-1}\left(\frac{t_{max}}{1-t_{max}}\right)$. Thus we get:

$$\frac{1-F_n(t_{max})}{1-f_n(t_{max})} \leq \left[\frac{1-\left(1-\frac{t_{max}}{1-t_{max}}\right)^{n-1}}{1-(1-t_{max})^n}\right]^{n-1} = G_n(t_{max})$$

Proof of Theorem 2.2: As we have already mentioned, for technical reasons, we shall divide the interval (0,1) into three parts, and we will examine separately the possibilities that t_{max} occurs in each of these parts.

More precisely, choose t_n such that $(1-t_n)^n = \frac{\log n}{n}$ and consider the intervals $(0, t_n)$, $\left[t_n, \frac{1}{2}\right]$ and $\left(\frac{1}{2}, 1\right)$.

t_n is something like $\frac{\log n - \log\log n}{n}$ and is certainly less than $\frac{\log n}{n}$.

Numerical evidence indicates that t_{max} is about $\frac{\log n}{n}$ but we eliminate the other intervals directly.

- We shall prove that for $t \in \left(\frac{1}{2}, 1\right)$, the function $\frac{1-F_n}{1-f_n}$ is decreasing and therefore, t_{max} does not occur in this open interval.

  It is quite easy to calculate that $F_n(t) = 1 - n(1-t)^n$ when $t \in \left(\frac{1}{2}, 1\right)$ by integrating (2.2) where $F_{n-1}\left(\frac{y}{1-y}\right) = 1$.

  So, $\frac{1-F_n}{1-f_n}$ becomes:

$$\frac{n(1-t)^n}{1-(1-(1-t)^n)^n}.$$

  If we put $s = 1-(1-t)^n$, we get,

$$\frac{1-F_n}{1-f_n} = \frac{n(1-s)}{1-s^n}$$

$$= \quad n(s^{n-1} + s^{n-2} + \cdots + 1)^{-1}$$

Which is a decreasing function of s, and therefore a decreasing function of t.

- We shall prove that for all t in $(0, t_n)$ not only is the function $\frac{1-F_n(t)}{1-f_n(t)}$ close to 1, but so is the function $(1 - f_n(t))^{-1}$.

  Since f_n is increasing,

$$\begin{aligned} f_n(t) &= (1-(1-t)^n)^n \\ &\leq (1-(1-t_n)^n)^n \\ &= \left(1 - \frac{\log n}{n}\right)^n \\ &\leq \exp(-\log n) = \frac{1}{n} \end{aligned}$$

  Hence,

$$\frac{1}{1-f_n} = 1 + O\left(\frac{1}{n}\right)$$

- Finally we study $F_n(t)$ for $t \in \left[t_n, \frac{1}{2}\right]$

  By Lemma 2.1,

$$\frac{1-F_n(t_{max})}{1-f_n(t_{max})} \leq G_n(t_{max})$$

  We shall prove that $G_n(t)$ is as small as required in the range $t \in \left[t_n, \frac{1}{2}\right]$, namely that

$$G_n(t) \leq 1 + O\left(\frac{(\log n)^3}{n}\right)$$

By the first part of Lemma 2.1, G_n in this range is:

$$G_n(t) = \left[1 + \frac{(1-t)^n - \left(1 - \frac{t}{1-t}\right)^{n-1}}{1-(1-t)^n}\right]^{n-1}$$

Thus, it is enough to prove that

$$\frac{(1-t)^n - \left(1 - \frac{t}{1-t}\right)^{n-1}}{1-(1-t)^n} \leq O\left(\frac{(\log n)^3}{n^2}\right)$$

Since the factor $1-(1-t)^n$ is like a constant in this interval, it suffices to show that $(1-t)^n - \left(1 - \frac{t}{1-t}\right)^{n-1}$ is dominated by the decreasing function $n(1-t)^{n-2}t^2$ (decreasing for $t \geq 2/n$) which at t_n is as small as required. Indeed,

$$\begin{aligned}
(1-t)^n - \left(1 - \frac{t}{1-t}\right)^{n-1} &\leq (1-t)^n - \left(1 - \frac{t}{1-t}\right)^{n} \\
&= \int_{1-\frac{t}{1-t}}^{1-t} nu^{n-1}du \\
&\leq \frac{t^2}{1-t}n(1-t)^{n-1} \\
&= n(1-t)^{n-2}t^2 \\
&\leq n(1-t_n)^{n-2}t_n^2 \\
&\leq 2n(1-t_n)^{n}t_n^2 \\
&\leq 2n\frac{\log n}{n}\frac{(\log n)^2}{n^2} \\
&= O\left(\frac{(\log n)^3}{n^2}\right)
\end{aligned}$$

which completes the proof.

Chapter 3

Coordinate slabs of the ℓ_p^n balls

In this Chapter we give a sketch of the proof of Theorem 2.1 for the general case $p \geq 1$.

As in the previous chapter, the subindependence property gives an upper bound for the proportion of the volume of the ℓ_p^n ball which is inside the cube $[-t, t]^n$.

Theorem 3.1, an estimate in the reverse direction which generalizes Theorem 2.2, states that this upper bound is again a very good approximation in the extreme cases when $\frac{n}{p}$ is either like zero or infinity.

A remark upon Theorem 3.1, shows that this may not be the case when $\frac{n}{p}$ is held fixed.

For the sake of readability, we are not going to introduce new notation with "p" subscripts, so we again use $F_n(t_1, \ldots, t_n)$ for the proportion of the volume of the ℓ_p^n ball which is inside the box $[-t_1, t_1] \times \cdots \times [-t_n, t_n]$ (or $F_n(t)$

when $t_1 = \cdots = t_n = t$), $Y_n(t)$ for $\int_0^t (1-u^p)^{(n-1)/p} du$ and $f_n(t)$ for $\left[\frac{Y_n(t)}{Y_n(1)}\right]^n$, the upper bound of $F_n(t)$, resulting from Theorem 2.1. By $v_{n,p}$ we denote the volume of the B_p^n ball.

3.1 The subindependence property

Sketch of the proof of Theorem 2.1 *for the case* $p > 1$,

Since the proof of this case does not differ too much from the one given for the case $p = 1$, we shall only write the two basic equations that are used in place of (2.8) and (2.9).

The relations are as follows;

$F_n(y, t_2, \ldots, t_n) =$

$$\begin{aligned} &= \frac{2v_{n-1,p}}{v_{n,p}} \int_0^y (1-u^p)^{\frac{n-1}{p}} F_{n-1}\left(\left(\frac{t_2^p}{1-u^p}\right)^{\frac{1}{p}}, \ldots, \left(\frac{t_n^p}{1-u^p}\right)^{\frac{1}{p}}\right) du \qquad (3.1) \\ &\leq \frac{2v_{n-1,p}}{v_{n,p}} F_{n-1}\left(\left(\frac{t_2^p}{1-y^p}\right)^{\frac{1}{p}}, \ldots, \left(\frac{t_n^p}{1-y^p}\right)^{\frac{1}{p}}\right) Y_n(y) \end{aligned}$$

$\frac{d}{dy} F_n(y, t_2, \ldots, t_n) =$

$$\begin{aligned} &= \frac{2v_{n-1,p}}{v_{n,p}} (1-y^p)^{\frac{n-1}{p}} F_{n-1}\left(\left(\frac{t_2^p}{1-y^p}\right)^{\frac{1}{p}}, \ldots, \left(\frac{t_n^p}{1-y^p}\right)^{\frac{1}{p}}\right) \qquad (3.2) \\ &= \frac{2v_{n-1,p}}{v_{n,p}} F_{n-1}\left(\left(\frac{t_2^p}{1-y^p}\right)^{\frac{1}{p}}, \ldots, \left(\frac{t_n^p}{1-y^p}\right)^{\frac{1}{p}}\right) \frac{d}{dy} Y_n(y) \end{aligned}$$

3.2 An estimate in the reverse direction

In this section we prove a result analogous to Theorem 2.2, but for the ℓ_p^n ball: namely a lower bound for the proportion of its volume which is inside the cube $[-t,t]^n$ (Theorem 3.1).

The method we follow, is more or less the same as the one we used in Theorem 2.2. But the process is carried out in full, because we want our results not only to generalise Theorem 2.2, but also to describe the way the proportion changes, when n and p change independently.

The Theorem we prove is:

Theorem 3.1 *For any n and p, the following estimates hold:*

I. If ϕ is such that $0<\phi\le \mathrm{e}^{-2}$ and t_n such that $(1-t_n^p)^{\frac{n}{p}}=\phi$, then,

$$\frac{1-F_n(t)}{1-f_n(t)} \le \begin{cases} \left[1-\left(1-\frac{\phi}{2p}\right)^n\right]^{-1} & \textit{for} \quad t\in(0,t_n) \\ \left[1+c\cdot\frac{1}{n}\cdot\phi\left(-\log\phi\right)^{1+1/p}\right]^{n-1} & \textit{for} \quad t\in[t_n,1) \end{cases}$$

where c is an absolute constant.

II.

$$\frac{1-F_n(t)}{1-f_n(t)} \le 2^{\frac{n-1}{p}}$$

A first obvious consequence of the second part of the Theorem 3.1 is the following Corollary.

Corollary 3.1.1

$$\frac{1-F_n(t)}{1-f_n(t)} \longrightarrow 1 \quad \textit{as} \quad \frac{p}{n}\longrightarrow\infty \quad \textit{uniformly in } t.$$

The result of Corollary 3.1.1, justifies our feeling that as p grows and the ℓ_p^n ball approaches more and more to the cube, F_n looks more like f_n. In other words, slabs behave more and more as if they are independent.

Because our initial aim was to generalize Theorem 2.2, we must also prove a result in which n tends to infinity. This is done in the following Corollary.

Corollary 3.1.2 *For each n and p, we define $\phi = \frac{2p}{n} \cdot \log\left(\frac{n\mathrm{e}^{-2}}{2p} + 1\right)$. Then for t_n such that $(1 - t_n^p)^{\frac{n}{p}} = \phi$,*

$$\frac{1 - F_n(t)}{1 - f_n(t)} \leq \begin{cases} 1 + c_1 \cdot \frac{p}{n} & \textit{for} \quad t \in (0, t_n) \\ \left[1 + c_2 \cdot \frac{1}{n} \cdot \phi\,(-\log\phi)^{1+1/p}\right]^{n-1} & \textit{for} \quad t \in [t_n, 1) \end{cases}$$

where c_1 and c_2 are universal constants.

As a consequence, we have that as $\frac{n}{p} \longrightarrow \infty$,

$$\frac{1 - F_n(t)}{1 - f_n(t)} = 1 + O\left(\frac{(\log(n/p + 1))^{2+1/p}}{n/p}\right)$$

uniformly in t.

Proof of Corollary 3.1.2: We first prove that $\phi \leq \mathrm{e}^{-2}$, so that we may invoke Theorem 3.1. Indeed,

$$\phi = \frac{2p}{n} \cdot \log\left(\frac{n\mathrm{e}^{-2}}{2p} + 1\right) \leq \frac{2p}{n} \cdot \frac{n\mathrm{e}^{-2}}{2p} = \mathrm{e}^{-2}$$

Then, according to Theorem 3.1, in the interval $(0, t_n)$

$$\frac{1 - F_n(t)}{1 - f_n(t)} \quad \leq \quad \left[1 - \left(1 - \frac{\phi}{2p}\right)^n\right]^{-1}$$

$$
\begin{aligned}
&= \left[1 - \left(1 - \frac{\log\left(\frac{n\mathrm{e}^{-2}}{2p} + 1\right)}{n}\right)^{n}\right]^{-1} \\
&\leq \left[1 - \left(\frac{n\mathrm{e}^{-2}}{2p} + 1\right)^{-1}\right]^{-1} \\
&= 1 + 2\mathrm{e}^{2} \cdot \frac{p}{n}
\end{aligned}
$$

Obviously, $\phi \leq 2\frac{\log(n/p+1)}{n/p}$.

Also when $\frac{n}{p}$ is large enough, $-\log\phi \leq \log(n/p+1)$, which explains the consequence.

By Corollary 3.1.1 and Corollary 3.1.2 we conclude that as $\frac{n}{p}$ tends to zero or infinity, $\frac{1-F_n}{1-f_n}$ tends to 1, uniformly in t. The natural thing to ask then, is what happens when $\frac{n}{p}$ tends to a positive number. Is it true that even so, $\frac{1-F_n}{1-f_n}$ tends to 1? The answer is no. A counterexample is given in the second Remark upon Theorem 3.1, appearing later on.

Our task now is to prove Theorem 3.1. As was mentioned above, the idea of the proof remains the same as in Theorem 2.2. We again focus our attention on the point t_{max}, where $\frac{1-F_n(t)}{1-f_n(t)}$ attains its maximum value. In the first Lemma below, we find a function $G_n(t)$ which dominates $\frac{1-F_n(t)}{1-f_n(t)}$ at t_{max}. This related function, is proved to be bounded as required in a particular range. Outside this range $\frac{1-F_n(t)}{1-f_n(t)}$ is small for very simple reasons.

The following Lemma, which gives this related function $G_n(t)$, is the analogue of Lemma 2.1.

Lemma 3.1 *At its maximum point, the function* $\frac{1-F_n(t)}{1-f_n(t)}$ *is dominated by the value of the function* $G_n(t) = \begin{cases} \left[\dfrac{Y_{n-1}\left(\left(\frac{t^p}{1-t^p}\right)^{\frac{1}{p}}\right)/Y_{n-1}(1)}{Y_n(t)/Y_n(1)}\right]^{n-1} & , \quad 0 < t^p \leq 1/2 \\ [Y_n(t)/Y_n(1)]^{-(n-1)} & , \quad 1/2 < t^p < 1 \end{cases}$

Proof of Lemma 3.1: The proof is very similar to the proof of Lemma 2.1, so it is omitted.

Before giving the proof of Theorem 3.1, we prove two Lemmata that will be used there. In the case of $p = 1$, these Lemmata become trivial, which is the reason they are not isolated in the argument for Theorem 2.2. In Lemma 3.2, we prove that

$$\frac{Y_{n-1}(t)}{Y_{n-1}(1)} \leq \frac{Y_n(t)}{Y_n(1)}$$

and in Lemma 3.3 we give a lower and an upper bound for t_n, the point defined in Theorem 3.1, which again determines the partition of (0,1) into three intervals, needed to study $\frac{1-F_n}{1-f_n}$ in a more effective way. These intervals are $(0, t_n)$, $[t_n, 2^{-1/p}]$ and $(2^{-1/p}, 1)$.

In the proof of Theorem 3.1 we use an approximate formula for $\frac{Y_n(t)}{Y_n(1)}$ which is proved in the Appendix in Lemma A.1. We do not discuss this in the main part of this section, because its proof is quite technical.

Lemma 3.2 *For* $t > 0$*, let* $S_n(t)$ *be the slab* $\{|x_1| \leq t\}$*. Then,*

$$\frac{Vol_n(S_n(t) \cap B_p^n)}{Vol_n(B_p^n)} \geq \frac{Vol_{n-1}(S_{n-1}(t) \cap B_p^{n-1})}{Vol_{n-1}(B_p^{n-1})}$$

Proof:

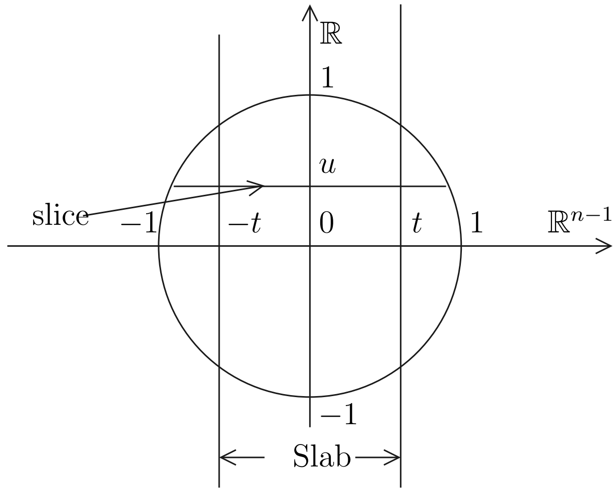

Figure 1

It is easy to see that at a level "$x_n = u$"(see Figure 1) the proportion of the volume of the $(n-1)$-dimensional ball which is inside the n-dimensional slab $S_n(t)$ at this level, is greater than the proportion of the volume of the unit ℓ_p^{n-1} ball which is inside the same slab. This tells us that,

$$\frac{Vol_{n-1}(S_n(t) \cap B_p^n \cap \{x_n = u\})}{(1-u^p)^{\frac{n-1}{p}} Vol_{n-1}(B_p^{n-1})} \geq \frac{Vol_{n-1}(S_{n-1}(t) \cap B_p^{n-1})}{Vol_{n-1}(B_p^{n-1})}.$$

Therefore,

$$\begin{aligned} Vol_n(S_n(t) \cap B_p^n) &= 2v_{n-1,p} \int_0^1 \frac{Vol_{n-1}(S_n(t) \cap B_p^n \cap \{x_n = u\})}{Vol_{n-1}(B_p^{n-1})} du \\ &\geq 2v_{n-1,p} \int_0^1 (1-u^p)^{\frac{n-1}{p}} \frac{Vol_{n-1}(S_{n-1}(t) \cap B_p^{n-1})}{Vol_{n-1}(B_p^{n-1})} du \\ &= \frac{Vol_{n-1}(S_{n-1}(t) \cap B_p^{n-1})}{Vol_{n-1}(B_p^{n-1})} Vol_n(B_p^n) \end{aligned}$$

In the next Lemma we give lower and upper bounds for t_n. It is clear from the proof why we need some condition like $\phi \leq \mathrm{e}^{-2}$, although we could replace it by the weaker one, $\phi \leq \left(\frac{n}{n+p+1}\right)^{\frac{n}{p}}$.

Lemma 3.3 *If ϕ is such that $0 < \phi \leq \mathrm{e}^{-2}$, then for t_n such that $(1-t_n^p)^{\frac{n}{p}} = \phi$,*

$$\frac{p+1}{n+p+1} \leq t_n^p \leq \frac{p(-\log\phi)}{n}$$

Proof:

The first inequality is equivalent to

$$\begin{aligned} (1-t_n^p)^{\frac{n}{p}} &\leq \left(\frac{n}{n+p+1}\right)^{\frac{n}{p}} \\ \Leftrightarrow \left(1+\frac{p+1}{n}\right)^{\frac{n}{p}} &\leq (1-t_n^p)^{-\frac{n}{p}} \\ \Leftrightarrow \left(1+\frac{p+1}{n}\right)^{\frac{n}{p}} &\leq \frac{1}{\phi} \end{aligned}$$

but the last inequality is obviously true since

$$\left(1+\frac{p+1}{n}\right)^{\frac{n}{p}} \leq \mathrm{e}^{\frac{p+1}{p}} \leq \mathrm{e}^2 \leq \frac{1}{\phi}$$

For the second one, if $\frac{p(-\log\phi)}{n} \geq 1$, we have nothing to prove. Otherwise, it is easy to see that

$$\left(1-\frac{p(-\log\phi)}{n}\right)^{\frac{n}{p}} \leq \phi = (1-t_n^p)^{\frac{n}{p}}$$

which implies what we want.

Proof of part I of Theorem 3.1: As we have already mentioned, for technical reasons, we shall divide the interval (0,1) into three parts, and we will examine separately the possibilities that t_{max} occurs in each of these parts. These are $(0, t_n)$, $\left[t_n, \left(\frac{1}{2}\right)^{\frac{1}{p}}\right]$ and $\left(\left(\frac{1}{2}\right)^{\frac{1}{p}}, 1\right)$.

- We shall prove that for $t^p \in \left(\frac{1}{2}, 1\right)$, the function $\frac{1-F_n(t)}{1-f_n(t)}$ is decreasing, so t_{max} is not in this open interval. Indeed, it is quite easy to calculate that in this interval, $1 - F_n(t) = n\left(1 - \frac{Y_n(t)}{Y_n(1)}\right)$. So, $\frac{1-F_n(t)}{1-f_n(t)}$ becomes

$$\frac{n\left(1 - \frac{Y_n(t)}{Y_n(1)}\right)}{1 - \left(\frac{Y_n(t)}{Y_n(1)}\right)^n}$$

  or if we put $s = \frac{Y_n(t)}{Y_n(1)}$,

$$\frac{n(1-s)}{1-s^n} = n(1 + s + s^2 + \cdots + s^{n-1})^{-1}$$

  which is a decreasing function in s, so it is a decreasing function in t as well.

- We shall prove that for all t in $(0, t_n)$ not only is the function $\frac{1-F_n(t)}{1-f_n(t)}$ bounded by the appropriate expression, but so is $(1 - f_n(t))^{-1}$.

  Since f_n is increasing,

$$f_n(t) = \left[\frac{Y_n(t)}{Y_n(1)}\right]^n \leq \left[\frac{Y_n(t_n)}{Y_n(1)}\right]^n$$

  Using Lemma A.1 to bound $\frac{Y_n(t_n)}{Y_n(1)}$, we get:

$$\begin{aligned} f_n(t) &\leq \left(1 - \frac{1}{2p}(1 - t_n^p)^{\frac{n}{p}}\right)^n \\ &= \left(1 - \frac{\phi}{2p}\right)^n \end{aligned}$$

- Finally we study $F_n(t)$ for $t \in \left[t_n, \left(\frac{1}{2}\right)^{1/p}\right]$

  By Lemma 3.1 ,

$$\frac{1 - F_n(t_{max})}{1 - f_n(t_{max})} \leq G_n(t_{max})$$

  We shall prove that $G_n(t)$ is as small as required in this range. Since

$$G_n(t) = \left[1 + \frac{Y_{n-1}\left(\left(\frac{t^p}{1-t^p}\right)^{\frac{1}{p}}\right)/Y_{n-1}(1) - Y_n(t)/Y_n(1)}{Y_n(t)/Y_n(1)}\right]^{n-1}$$

  it suffices to prove that

$$\frac{Y_{n-1}\left(\left(\frac{t^p}{1-t^p}\right)^{\frac{1}{p}}\right)/Y_{n-1}(1) - Y_n(t)/Y_n(1)}{Y_n(t)/Y_n(1)} \leq O\left(\frac{1}{n} \cdot \phi\left(-\log\phi\right)^{1+1/p}\right)$$

  But in this interval, the denominator $\frac{Y_n(t)}{Y_n(1)}$ is like a constant since by Lemma A.1

$$\frac{Y_n(t)}{Y_n(1)} \geq \frac{Y_n(t_n)}{Y_n(1)} \geq 1 - (1 - t_n^p)^{\frac{n}{p}} = 1 - \phi \geq 1 - \mathrm{e}^{-2}$$

  So it is enough to show that

$$\frac{Y_{n-1}\left(\left(\frac{t^p}{1-t^p}\right)^{\frac{1}{p}}\right)}{Y_{n-1}(1)} - \frac{Y_n(t)}{Y_n(1)} \leq O\left(\frac{1}{n} \cdot \phi\left(-\log\phi\right)^{1+1/p}\right)$$

  But, by Lemma 3.2 we get:

$$\begin{aligned}
\frac{Y_{n-1}\left(\left(\frac{t^p}{1-t^p}\right)^{\frac{1}{p}}\right)}{Y_{n-1}(1)} - \frac{Y_n(t)}{Y_n(1)} &\leq \frac{Y_n\left(\left(\frac{t^p}{1-t^p}\right)^{\frac{1}{p}}\right)}{Y_n(1)} - \frac{Y_n(t)}{Y_n(1)} \\
&= \frac{1}{Y_n(1)}\int_t^{\left(\frac{t^p}{1-t^p}\right)^{\frac{1}{p}}} (1-u^p)^{\frac{n-1}{p}} du \\
&\leq \frac{1}{Y_n(1)}(1-t^p)^{\frac{n-1}{p}}\left(\left(\frac{t^p}{1-t^p}\right)^{\frac{1}{p}} - t\right) \\
&\leq \frac{1}{Y_n(1)}\frac{t^{p+1}}{p(1-t^p)}(1-t^p)^{\frac{n-1}{p}}
\end{aligned}$$

The last inequality is true because

$$\begin{aligned}
\frac{t^{2p}}{1-t^p} &= \frac{t^p}{1-t^p} - t^p \\
&= \int_t^{\left(\frac{t^p}{1-t^p}\right)^{\frac{1}{p}}} p x^{p-1} dx \\
&\geq p t^{p-1} \left(\left(\frac{t^p}{1-t^p} \right)^{\frac{1}{p}} - t \right)
\end{aligned}$$

Now we notice that in this interval, the function $t^{p+1}(1-t^p)^{\frac{n}{p}}$ is decreasing. (It is decreasing for $t^p \geq \frac{p+1}{n+p+1}$ which is a wider range according to Lemma 3.3). Also, $(1-t^p)^{-1} \leq 2$.

So we have:

$$\begin{aligned}
\frac{Y_{n-1}\left(\left(\frac{t^p}{1-t^p}\right)^{\frac{1}{p}}\right)}{Y_{n-1}(1)} - \frac{Y_n(t)}{Y_n(1)} &\leq \frac{4}{pY_n(1)} t^{p+1}(1-t^p)^{\frac{n}{p}} \\
&\leq \frac{4}{pY_n(1)} t_n^{p+1}(1-t_n^p)^{\frac{n}{p}} \\
&= \frac{4}{pY_n(1)} t_n^{p+1} \phi
\end{aligned}$$

To complete the proof, we only need apply Lemma 3.3, in order to bound t_n by $\frac{p(-\log \phi)}{n}$, and to notice that $Y_n(1)$ is like $(1/n)^{1/p}$. Thus,

$$\begin{aligned}
\frac{Y_{n-1}\left(\left(\frac{t^p}{1-t^p}\right)^{\frac{1}{p}}\right)}{Y_{n-1}(1)} - \frac{Y_n(t)}{Y_n(1)} &\leq c \cdot \frac{4}{p} \cdot n^{1/p} \left(\frac{p(-\log \phi)}{n}\right)^{1+1/p} \phi \\
&= c \cdot \frac{1}{n} \cdot \phi(-\log \phi)^{1+1/p}
\end{aligned}$$

Where c is an absolute constant.

Proof of part II of Theorem 3.1: Again we focus our attention at t_{max}, the point where $\frac{1-F_n}{1-f_n}$ attains its maximum. We shall prove that

$$\frac{1-F_n(t_{max})}{1-f_n(t_{max})} \leq 2^{\frac{n-1}{p}}$$

.

Since $\frac{1-F_n(t_{max})}{1-f_n(t_{max})} \leq G_n(t_{max})$ (Lemma 3.1) and t_{max} does not belong in $(2^{-1/p}, 1)$, as was observed in the proof of the first part of Theorem 3.1, it suffices to prove that for all t in $(0, 2^{-1/p}]$,

$$G_n(t) \leq 2^{\frac{n-1}{p}}$$

Using Lemma 3.1 and Lemma 3.2, for $t \in (0, 2^{-1/p}]$,

$$G_n(t) \leq \left[\frac{Y_n\left(\left(\frac{t^p}{1-t^p}\right)^{1/p} \right)}{Y_n(t)} \right]^{n-1} \tag{3.3}$$

Substituting $x = (1-t^p)^{1/p}\, u$ in $\int_0^{\left(\frac{t^p}{1-t^p}\right)^{1/p}} (1-u^p)^{\frac{n-1}{p}} du$ we get:

$$\begin{aligned}
Y_n\left(\left(\frac{t^p}{1-t^p} \right)^{1/p} \right) &= \int_0^{\left(\frac{t^p}{1-t^p}\right)^{1/p}} (1-u^p)^{\frac{n-1}{p}} du \\
&= \int_0^t \left(1 - \frac{x^p}{1-t^p} \right)^{\frac{n-1}{p}} (1-t^p)^{-1/p} dx \\
&= (1-t^p)^{-\frac{n}{p}} \int_0^t (1 - x^p - t^p)^{\frac{n-1}{p}} dx \\
&\leq (1-t^p)^{-\frac{n}{p}} \int_0^t [(1-x^p)(1-t^p)]^{\frac{n-1}{p}} dx \\
&= (1-t^p)^{-\frac{1}{p}} \int_0^t (1-x^p)^{\frac{n-1}{p}} dx \\
&= (1-t^p)^{-\frac{1}{p}} Y_n(t)
\end{aligned}$$

Substituting this in (3.3), we get that

$$G_n(t) \leq (1-t^p)^{-\frac{n-1}{p}} \leq 2^{\frac{n-1}{p}}$$

Remarks

1. In the proof of the first part of Theorem 3.1, we have actually proved that for $t \in [t_n, 1)$,

$$\frac{1 - F_n(t)}{1 - f_n(t)} \leq \left[1 + c \cdot \frac{1}{n} \cdot \phi(-\log \phi)^{1+1/p}\right]^{n-1}$$

2. As was mentioned in the begining of this section, by the Corollaries 3.1.1 and 3.1.2, we can conclude that $\frac{1-F_n}{1-f_n} \longrightarrow 1$ as $\frac{n}{p} \longrightarrow 0$ or ∞. The natural thing to ask then, is whether we can have a similar statement even when $\frac{n}{p}$ converges to a positive number. The answer is negative, and we give a counterexample:

   Take the case where $p = n - 1$. In this case we can easily calculate that

$$\frac{Y_n(t)}{Y_n(1)} = \frac{n+1}{n}\left(1 - \frac{t^n}{n+1}\right)t$$

   Therefore for $t = 2^{-1/p} = 2^{-1/(n-1)}$ and n large enough we have:

$$\left(\frac{Y_n(t)}{Y_n(1)}\right)^n \quad \sim \quad \mathrm{e} \cdot \mathrm{e}^{-1/2} \cdot \frac{1}{2} \quad \leq \quad \frac{7}{8}$$

   Thus,

$$\frac{Y_n(t)}{Y_n(1)} \leq \left(\frac{7}{8}\right)^{1/n}$$

   But for $t \in [2^{-1/p}, 1)$ we know the precise formula for $\frac{1-F_n(t)}{1-f_n(t)}$ which is a decreasing function of $\frac{Y_n(t)}{Y_n(1)}$ as was observed in the proof of Theorem 3.1. So, for $t = 2^{-1/p}$ and n large enough (and therefore p large enough), we have:

$$\begin{aligned}
\frac{1-F_n(t)}{1-f_n(t)} &= \frac{n\left(1-\frac{Y_n(t)}{Y_n(1)}\right)}{1-\left(\frac{Y_n(t)}{Y_n(1)}\right)^n} \\
&\geq \frac{n\left(1-\left(\frac{7}{8}\right)^{1/n}\right)}{1-\frac{7}{8}} \\
&= 8n\left(1-\left(\frac{7}{8}\right)^{1/n}\right) \\
&\geq 1.01
\end{aligned}$$

The last inequality is true, simply because for n large,

$$\left(1-\frac{1.01}{8n}\right)^n \sim \exp\left(-\frac{1.01}{8}\right) > \frac{7}{8}$$

.

Chapter 4

Complements of coordinate slabs

The subindependence of coordinate slabs discussed in the previous chapters, made use of an accurate method for studying volumes. Here, using exactly the same ideas, we prove a complementary result, the subindependence of the complements of coordinate slabs, stated in the Theorem below.

It is actually this statement that will be used in the last Chapter where we prove a sort of Central Limit Theorem.

The proof, though similar to the one of Theorem 2.1, is given in full, but not in great detail, as we assume the reader has become familiar with the method.

Theorem 4.1 (Subindependence of complements of coordinate slabs) *If the probability P is normalised Lebesgue measure on one of the ℓ_p^n balls in $\mathbf{R}^n$, then for any sequence $t_1, \ldots, t_n$ of positive numbers,*

$$P\left(\cap_1^n \{|x_i| \geq t_i\}\right) \leq \prod_1^n P(\{|x_i| \geq t_i\}).$$

Proof : For a fixed $p \geq 1$, let $\widehat{F}_n(t_1, \ldots, t_n)$ denote the proportion of the volume of the unit ℓ_p^n ball which is outside of all slabs $\{|x_i| < t_i\}$, and $\widehat{Y}_n(t)$ the integral $\int_{min\{1,t\}}^1 (1-u^p)^{\frac{n-1}{p}} du$. Then, Theorem 4.1 states that

$$\widehat{F}_n(t_1, \ldots, t_n) \leq \frac{\widehat{Y}_n(t_1)}{\widehat{Y}_n(0)} \cdots \frac{\widehat{Y}_n(t_n)}{\widehat{Y}_n(0)}$$

The case that at least one of the t_i's is greater than or equal to 1 is trivial.

Suppose then that this is not the case. We shall prove that the value of the function $\widehat{F}_n$ at point $(t_1, \ldots, t_n)$ is dominated by an appropriate multiple of the value of $\widehat{F}_n$, at the point with the ith coordinate replaced by 0 and the rest remaining the same, i.e.

$$\widehat{F}_n(t_1, \ldots, t_n) \leq \frac{\widehat{Y}_n(t_i)}{\widehat{Y}_n(0)} \widehat{F}_n(t_1 \ldots, t_{i-1}, 0, t_{i+1} \ldots, t_n) \tag{4.1}$$

So, we shall have in turn the following inequalities:

$$\begin{aligned}
\widehat{F}_n(t_1, \ldots, t_n) &\leq \frac{\widehat{Y}_n(t_1)}{\widehat{Y}_n(0)} \widehat{F}_n(0, t_2 \ldots, t_n) \\
&\leq \frac{\widehat{Y}_n(t_1)}{\widehat{Y}_n(0)} \frac{\widehat{Y}_n(t_2)}{\widehat{Y}_n(0)} \widehat{F}_n(0, 0, t_3 \ldots, t_n) \\
&\quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&\leq \frac{\widehat{Y}_n(t_1)}{\widehat{Y}_n(0)} \cdots \frac{\widehat{Y}_n(t_n)}{\widehat{Y}_n(0)} \widehat{F}_n(0, \ldots, 0)
\end{aligned}$$

Since $\widehat{F}_n(0,\ldots,0)=1$, the proof is complete.

Thus, the crucial point is to prove (4.1). Without loss of generality, we shall prove this for $i=n$, namely the relation:

$$\widehat{F}_n(t_1,\ldots,t_n) \leq \frac{\widehat{Y}_n(t_n)}{\widehat{Y}_n(0)}\widehat{F}_n(t_1\ldots,t_{n-1},0) \tag{4.2}$$

To do this, we combine two equations. The first one relates $\widehat{F}_n$ and $\widehat{F}_{n-1}$, and the second one relates $\widehat{F}_{n-1}$ and the partial derivative of $\widehat{F}_n$ with respect to the n-th coordinate.

These are:

$$\begin{aligned}
&\widehat{F}_n(t_1,\ldots,t_{n-1},x) = \\
&\quad = \frac{2v_{n-1,p}}{v_{n,p}}\int_x^1 (1-u^p)^{\frac{n-1}{p}}\widehat{F}_{n-1}\left(\frac{t_1}{(1-u^p)^{1/p}},\ldots,\frac{t_{n-1}}{(1-u^p)^{1/p}}\right)du \qquad (4.3)\\
&\quad \leq \frac{2v_{n-1,p}}{v_{n,p}}\widehat{F}_{n-1}\left(\frac{t_1}{(1-x^p)^{1/p}},\ldots,\frac{t_{n-1}}{(1-x^p)^{1/p}}\right)\int_x^1 (1-u^p)^{\frac{n-1}{p}}du\\
&\quad = \frac{2v_{n-1,p}}{v_{n,p}}\widehat{F}_{n-1}\left(\frac{t_1}{(1-x^p)^{1/p}},\ldots,\frac{t_{n-1}}{(1-x^p)^{1/p}}\right)\widehat{Y}_n(x)
\end{aligned}$$

and

$$\begin{aligned}
&\tfrac{d}{dx}\widehat{F}_n(t_1,\ldots,t_{n-1},x) = \\
&\quad = -\frac{2v_{n-1,p}}{v_{n,p}}(1-x^p)^{\frac{n-1}{p}}\widehat{F}_{n-1}\left(\frac{t_1}{(1-x^p)^{1/p}},\ldots,\frac{t_{n-1}}{(1-x^p)^{1/p}}\right) \qquad (4.4)\\
&\quad = \frac{2v_{n-1,p}}{v_{n,p}}\widehat{F}_{n-1}\left(\frac{t_1}{(1-x^p)^{1/p}},\ldots,\frac{t_{n-1}}{(1-x^p)^{1/p}}\right)\frac{d}{dx}\widehat{Y}_n(x)
\end{aligned}$$

where $v_{n,p}$ is the volume of the unit ℓ_p^n ball and x is a non-negative number, less than 1.

By eliminating the factor $\frac{2v_{n-1,p}}{v_{n,p}}\widehat{F}_{n-1}\left(\frac{t_1}{(1-x^p)^{1/p}},\ldots,\frac{t_{n-1}}{(1-x^p)^{1/p}}\right)$ we get:

$$\frac{\frac{d}{dx}\widehat{F}_n(t_1,\ldots,t_{n-1},x)}{\widehat{F}_n(t_1,\ldots,t_{n-1},x)} \leq \frac{\frac{d}{dx}\widehat{Y}_n(x)}{\widehat{Y}_n(x)} \tag{4.5}$$

and, by integrating from 0 to t_n we get (4.2).

It remains to prove (4.3) and (4.4).

For the first one we have:

$$
\begin{aligned}
&\widehat{F}_n(t_1,\ldots,t_{n-1},x) = \\
&= \frac{2}{v_{n,p}}\int_x^1 Vol_{n-1}(\{|x_i|\geq t_i, i=1,\ldots,n-1\}\cap B_p^n\cap\{x_n=u\})du\\
&= \frac{2}{v_{n,p}}\int_x^1 Vol_{n-1}\left(\{|x_i|\geq t_i, i=1,\ldots,n-1\}\cap B_p^{n-1}\left((1-u^p)^{1/p}\right)\right)du\\
&= \frac{2v_{n-1,p}}{v_{n,p}}\int_x^1 (1-u^p)^{\frac{n-1}{p}}\\
&\quad \frac{Vol_{n-1}\left(\{|x_i|\geq t_i, i=1,\ldots,n-1\}\cap B_p^{n-1}\left((1-u^p)^{1/p}\right)\right)}{Vol_{n-1}\left(B_p^{n-1}\left((1-u^p)^{1/p}\right)\right)}du\\
&= \frac{2v_{n-1,p}}{v_{n,p}}\int_x^1 (1-u^p)^{\frac{n-1}{p}}\widehat{F}_{n-1}\left(\frac{t_1}{(1-u^p)^{1/p}},\ldots,\frac{t_{n-1}}{(1-u^p)^{1/p}}\right)du
\end{aligned}
$$

To prove the second one, let

$$\widehat{H}_n(x) = Vol_n(\{|x_i|\geq t_i, i=1,\ldots,n-1, |x_n|\geq x\}\cap B_p^n)$$

Then,

$$\frac{d}{dx}\widehat{F}_n(t_1,\ldots,t_{n-1},x) = \frac{\frac{d}{dx}\widehat{H}_n(x)}{v_{n,p}}$$

But

$$
\begin{aligned}
&\tfrac{d}{dx}\widehat{H}_n(x) = -\lim_{h\to 0}\tfrac{\widehat{H}_n(x)-\widehat{H}_n(x+h)}{h} = \\
&\quad = -2Vol_{n-1}(\{|x_i|\geq t_i, i=1,\ldots,n-1\}\cap B_p^n\cap\{x_n=x\})\\
&\quad = -2Vol_{n-1}\left(\{|x_i|\geq t_i, i=1,\ldots,n-1\}\cap B_p^{n-1}\left((1-x^p)^{1/p}\right)\right)
\end{aligned}
$$

So,

$$
\begin{aligned}
&\tfrac{d}{dx}\widehat{F}_n(t_1,\dots,t_{n-1},x) = \\
&\quad = \; -\frac{2v_{n-1,p}}{v_{n,p}}(1-x^p)^{\frac{n-1}{p}} \times \\
&\qquad \times \frac{Vol_{n-1}\left(\{|x_i| \geq t_i, i=1,\dots,n-1\} \cap B_p^{n-1}\left((1-x^p)^{1/p}\right)\right)}{Vol_{n-1}\left(B_p^{n-1}\left((1-x^p)^{1/p}\right)\right)} \\
&\quad = \; -\frac{2v_{n-1,p}}{v_{n,p}}(1-x^p)^{\frac{n-1}{p}}\widehat{F}_{n-1}\left(\frac{t_1}{(1-x^p)^{1/p}},\dots,\frac{t_{n-1}}{(1-x^p)^{1/p}}\right)
\end{aligned}
$$

Chapter 5

A counterexample

Whitney and Loomis, proved an inequality which gives an upper bound for the volume of a convex body, in terms of the $(n-1)$-Lebesgue measures of its orthogonal projections onto the 1-codimensional subspaces perpendicular to an orthonormal basis. The Theorem they prove is:

Theorem 5.1 (Whitney-Loomis) *For any convex body K, if P_i is the orthogonal projection to $\langle e_i \rangle^{\perp}$, then,*

$$[Vol_n(K)]^{n-1} \leq \prod_{i=1}^{n} Vol_{n-1}(P_i(K))$$

There is a relation between this Theorem and Theorem 2.1, when K is one of the unit ℓ_p^n balls. In fact, we notice that if we rewrite Theorem 2.1 as

$$[Vol_n(K)]^{n-1} \leq \frac{\prod_{i=1}^{n} Vol_n(K \cap \{|x_i| \leq t\})}{Vol_n(K \cap Q_n(t))}$$

and then take limits for $t \to 0$, we get the Whitney-Loomis inequality for these particular K's.

Of course, in the above limit we don't actually take the orthogonal projections but instead we take the intersections of the body and the coordinate hyperplanes, which is the same since the body is coordinate symmetric.

Thus, in the case of the ℓ_p^n balls, the Whitney-Loomis Theorem is the limit case of Theorem 2.1. Since there is such a relation between the two Theorems, and since Whitney-Loomis Theorem is true for all convex bodies, it is natural to ask whether Theorem 2.1 is true of all coordinate symmetric convex bodies, not just the ℓ_p^n balls. It can be proved, by rather easy means, that this is true for $n = 2$. However, the answer is no, for $n \geq 3$.

We give a counterexample in the 3-dim Euclidean space. The coordinate symmetric convex body we consider, is an ℓ_1 ball, which has been stretched and rotated by 45° about the z-axis. The body is the convex hull of the points $(0, 0, \pm 1)$ and the square in the x, y plane with corners $(\pm 1, \pm 1, 0)$. This is:

$$K = co\{(1, 1, 0), (1, -1, 0), (-1, -1, 0), (-1, 1, 0), (0, 0, 1), (0, 0, -1)\}$$

Now, it is a simple calculation, to see that Theorem 2.1 fails in this case.

Indeed, let S_1 be the central slab perpendicular to the x-axis of width 1, i.e. $S_1 = \left\{|x_1| \leq \frac{1}{2}\right\}$, S_2 be the central slab perpendicular to the y-axis of width 1, i.e. $S_2 = \left\{|x_2| \leq \frac{1}{2}\right\}$, and S_3 be the central slab perpendicular to the z-axis of width 2 (which includes the body), i.e. $S_3 = \{|x_3| \leq 1\}$.

Then, $Vol\,(S_1 \cap K) = Vol\,(S_2 \cap K) = 11/6$, $Vol\,(S_3 \cap K) = Vol(K) = 8/3$ and $Vol\,(K \cap S_1 \cap S_2 \cap S_3) = 4/3$. So,

$$\frac{Vol\left(K \cap \left(\cap_{i=1}^{3} S_i\right)\right)}{Vol(K)} = \frac{1}{2}$$

and

$$\frac{Vol\left(K \cap S_1\right)}{Vol(K)} \cdot \frac{Vol\left(K \cap S_2\right)}{Vol(K)} \cdot \frac{Vol\left(K \cap S_3\right)}{Vol(K)} = \frac{121}{256} < \frac{1}{2}.$$

Chapter 6

A Central Limit Theorem

This section is almost entirely an application of the subindependence of coordinate slabs proved in the previous section. Our aim is to prove a sort of Central Limit Theorem for the ℓ_p balls. On the probability space of the normalized ℓ_p^n ball, say K, with probability measure the Lebesgue measure in K, we define the random variables $x \mapsto \langle x, \theta \rangle$, for each $\theta \in S^{n-1}$. We prove that the average of their densities is very close to a Gaussian.

More precisely, if we denote by $g_\theta(t)$ these densities, we prove that as n tends to infinity,

$$\int_{S^{n-1}} g_\theta(t) d\sigma(\theta) \quad \longrightarrow \quad \frac{1}{\varrho\sqrt{2\pi}} \exp\left(-\frac{t^2}{2\varrho^2}\right)$$

for all $t \in \mathbf{R}$.

Here, σ is the rotation invariant probability measure on the sphere $S^{n-1} = \{x \in \mathbf{R}^n : \sum x_i^2 = 1\}$, and ϱ is a number to be specified.

The proof is composed of two main steps. Firstly, we prove that

$$\int_{S^{n-1}} g_\theta(t) d\sigma(\theta) \sim \int_K \frac{1}{\sqrt{2\pi}} \frac{\sqrt{n}}{|x|} \exp\left(-\frac{t^2}{2} \frac{n}{|x|^2}\right)$$

where the symbol "$\sim$" is used here in the usual way:

$$f_n \sim g_n \quad \Leftrightarrow \quad \lim_{n\to\infty} \frac{f_n}{g_n} = 1.$$

Then, using the subindependence of the coordinate slabs and standard probabilistic arguments, we prove that "most" of the mass of K is inside a shell where $\frac{|x|^2}{n}$ is approximately ϱ^2. Applying this to the integral on the right hand side of the above relation, we get what is required.

6.1 Preliminaries

Before stating our Theorem we make a few remarks about these R.V.s and calculate their densities g_θ and their variances.

- First of all we notice that for each fixed n and p, they all have the same variance $\varrho_n = \int_K \langle x, \theta \rangle^2$.

  Indeed, write $\theta = \theta_1 \mathrm{e}_1 + \ldots + \theta_n \mathrm{e}_n$, where $\theta_i \in \mathbf{R}$ s.th. $\sum_{i=1}^n \theta_i^2 = 1$. Since K is coordinate symmetric, $\int_K \langle x, \mathrm{e}_i \rangle \langle x, \mathrm{e}_j \rangle = 0$ when $i \neq j$, and $\int_K \langle x, \mathrm{e}_i \rangle^2$ does not depend on i. Therefore, if we put $\varrho_n^2 = \int_K \langle x, \mathrm{e}_i \rangle^2$, we have:

$$\int_K \langle x, \theta \rangle^2 \;=\; \int_K \left(\sum_{i=1}^n \theta_i^2 \langle x, \mathrm{e}_i \rangle^2 + 2 \sum_{i \neq j} \theta_i \theta_j \langle x, \mathrm{e}_i \rangle \langle x, \mathrm{e}_j \rangle \right)$$

$$
\begin{aligned}
&= \sum_{i=1}^{n}\left(\theta_i^2\int_K \langle x, \mathrm{e}_i\rangle^2\right) + 2\sum_{i\neq j}\left(\theta_i\theta_j\int_K \langle x, \mathrm{e}_i\rangle\langle x, \mathrm{e}_j\rangle\right)\\
&= \sum_{i=1}^{n}\left(\theta_i^2\int_K \langle x, \mathrm{e}_i\rangle^2\right)\\
&= \sum_{i=1}^{n}\left(\theta_i^2\varrho_n^2\right)\\
&= \varrho_n^2
\end{aligned}
$$

- The sequence $\{\varrho_n\}_{n=1}^{\infty}$ converges to a number say ϱ. (This is the number that appears in the statement of the Central Limit Theorem above). Indeed, if we put λ for the radius of K, we get:

$$
\begin{aligned}
\varrho_n^2 = \int_K x_1^2 &= 2\int_0^{\lambda} u^2 Vol_{n-1}(K\cap\{x_1=u\})du\\
&= 2\int_0^{\lambda} u^2(\lambda^p-u^p)^{\frac{n-1}{p}} Vol_{n-1}(B_p^{n-1})du\\
&= 2v_{n-1,p}\int_0^1 \lambda^2 v^2\lambda^{n-1}(1-v^p)^{\frac{n-1}{p}}\lambda dv\\
&= 2\lambda^{n+2}v_{n-1,p}\int_0^1 v^2(1-v^p)^{\frac{n-1}{p}}dv\\
&= \frac{2v_{n-1,p}}{(v_{n,p})^{1+2/n}}\int_0^1 v^2(1-v^p)^{\frac{n-1}{p}}dv
\end{aligned}
$$

Using the fact that $v_{n,p} = \frac{[2\Gamma(1+1/p)]^n}{\Gamma(1+n/p)}$, and Stirling's formula, we get that $n^{-3/p}\frac{2v_{n-1,p}}{(v_{n,p})^{1+2/n}}$ converges to a constant c_p, which depends only on p. Actually, this c_p is bounded in "p", a piece of information which will be of use later.

But

$$
\begin{aligned}
\lim_{n\to\infty} n^{\frac{3}{p}}\int_0^1 v^2(1-v^p)^{\frac{n-1}{p}}dv &= \lim_{n\to\infty}\int_0^{n^{1/p}} y^2\left(1-\frac{y^p}{n}\right)^{\frac{n-1}{p}}dy\\
&= \int_0^{\infty} y^2\exp\left(-\frac{y^p}{p}\right)dy
\end{aligned}
$$

- The densities are the functions: $g_\theta(t) = Vol_{n-1}\left(K \cap \left(\langle\theta\rangle^\perp + t\theta\right)\right)$, since

$$\int_{-\infty}^{t} g_\theta(u)du = P(\langle x,\theta\rangle \le t) = \int_{-\infty}^{t} Vol_{n-1}\left(K \cap \left(\langle\theta\rangle^\perp + u\theta\right)\right) du$$

6.2 The basic approximation

We start by proving the critical approximation mentioned before, for the integral $\int_{S^{n-1}} g_\theta(t)d\sigma(\theta)$ that interests us. So, we want to prove that:

$$\int_{S^{n-1}} g_\theta(t)d\sigma(\theta) \sim \int_K \frac{1}{\sqrt{2\pi}}\frac{\sqrt{n}}{|x|}\exp\left(-\frac{t^2}{2}\frac{n}{|x|^2}\right) \tag{6.1}$$

We shall prove this for $t \ge 0$. A similar argument can be applied for $t \le 0$.

We first recall that if v is a unit vector in $\mathbf{R}^n$, then,

$$\sigma(|\langle\theta,v\rangle| > t) = \frac{\int_t^1 (1-u^2)^{\frac{n-3}{2}}du}{\int_0^1 (1-u^2)^{\frac{n-3}{2}}du}$$

So we have:

$$\begin{aligned}
\int_{S^{n-1}} g_\theta(t)d\sigma(\theta) &= \int_{S^{n-1}} \lim_{\delta\to 0}\frac{1}{\delta}Vol_n(K\cap\{t\le\langle x,\theta\rangle\le t+\delta\})d\sigma(\theta)\\
&= \lim_{\delta\to 0}\frac{1}{\delta}\int_{S^{n-1}}\int_K \mathbf{1}_{\{t\le\langle x,\theta\rangle\le t+\delta\}}dx\; d\sigma(\theta)\\
&= \lim_{\delta\to 0}\frac{1}{\delta}\int_K\int_{S^{n-1}} \mathbf{1}_{\{t\le\langle x,\theta\rangle\le t+\delta\}}d\sigma(\theta)\; dx\\
&= \lim_{\delta\to 0}\frac{1}{\delta}\int_K \sigma\left(\left\{\frac{t}{|x|}\le\left\langle\frac{x}{|x|},\theta\right\rangle\le\frac{t+\delta}{|x|}\right\}\right)dx\\
&= \lim_{\delta\to 0}\frac{1}{\delta}\int_K \frac{1}{2}\frac{\int_{t/|x|}^{(t+\delta)/|x|}(1-u^2)^{\frac{n-3}{2}}du}{\int_0^1(1-u^2)^{\frac{n-3}{2}}du}dx
\end{aligned}$$

We need only notice now that

$$\lim_{\delta\to 0}\frac{1}{\delta}\int_{t/|x|}^{(t+\delta)/|x|}(1-u^2)^{\frac{n-3}{2}}du = \frac{1}{|x|}\left(1-\frac{t^2}{|x|^2}\right)^{\frac{n-3}{2}} \sim \frac{1}{|x|}\exp\left(-\frac{t^2}{2}\frac{n}{|x|^2}\right)$$

and that

$$2\int_0^1 (1-u^2)^{\frac{n-3}{2}} du \sim \sqrt{\frac{n}{2\pi}}$$

to complete the proof of (6.1).

6.3 The main Theorem

The above integral over K is an average of Gaussian densities of different variances. The key idea is that most of these Gaussians have about the same variance because $\frac{|x|}{\sqrt{n}}$ is typically very close to ϱ_n: i.e. that the set where $\frac{|x|}{\sqrt{n}}$ is "far" from ϱ_n, has small probability. More precisely, it is proved in Lemma 6.3, that for all positive numbers r,

$$P\left(\left|\frac{|x|^2}{n} - \varrho_n^2\right| \geq r\right) \leq \frac{35}{nr^2}\varrho_n^4$$

Both Lemma 6.1, and the Corollary following Lemma 6.2 below, are used in the proof of Lemma 6.3. Our aim in these two statements, is to bound from above, expressions of the form $\int_K x_1^{2m_1} \dots x_l^{2m_l}$ by an appropriate expression involving just the familiar $\int_K x_i^2$ where x_i denotes $\langle x, \mathrm{e}_i \rangle$.

This is done in two steps:

Firstly, in Lemma 6.1 we prove a subindependence property for the coordinate R.V.s x_i. i.e. that:

$$\int_K x_1^{2m_1} \dots x_l^{2m_l} \leq \prod_{i=1}^{l} \int_K x_i^{2m_i}$$

Then in the Corollary to Lemma 6.2, we prove that each term of the form $\int_K x_i^{2m}$ can be bounded from above by a multiple of the m-th power of $\int_K x_i^2$. i.e. that:

$$\int_K x_i^{2m} \leq \frac{(2m)!}{2} 3^{m-1} \left(\int_K x_i^2 \right)^m$$

As was mentioned above, Lemma 6.1 uses the subindependence of the complements of the coordinate slabs of the ℓ_p^n-ball, while Lemma 6.2 uses standard results concerning log-concave functions.

Lemma 6.1 *With K a normalized ℓ_p-ball as above,*

$$\int_K x_1^{2m_1} \dots x_l^{2m_l} \leq \prod_{i=1}^{l} \int_K x_i^{2m_i}$$

Where the m_i are positive integers and $1 \leq l \leq n$.

Proof: We shall give the proof only for $l = 2$. The general case is similar (and in fact we only need the special case). In this case, we have to prove the inequality:

$$\int_K x_1^{2m_1} x_2^{2m_2} \leq \int_K x_1^{2m_1} \int_K x_2^{2m_2}$$

Notice that:

$$\int_{K \cap \{x_1 \geq 0, x_2 \geq 0\}} x_1^{2m_1} x_2^{2m_2} =$$

$$\begin{aligned}
&= \int_{K \cap \{x_1 \geq 0, x_2 \geq 0\}} \left(2m_1 \int_0^{x_1} u^{2m_1 - 1} du \right) \left(2m_2 \int_0^{x_2} v^{2m_2 - 1} dv \right) \\
&= 4m_1 m_2 \int_{K \cap \{x_1 \geq 0, x_2 \geq 0\}} \left(\int_{\mathbf{R}_+^2} u^{2m_1 - 1} v^{2m_2 - 1} \mathbf{1}_{\{x_1 \geq u, x_2 \geq v\}} du dv \right)
\end{aligned}$$

$$= \quad 4m_1m_2 \int_{\mathbf{R}_+^2} \left(\int_{K\cap\{x_1\geq 0, x_2\geq 0\}} u^{2m_1-1} v^{2m_2-1} \mathbf{1}_{\{x_1\geq u, x_2\geq v\}} \right) dudv$$

$$= \quad 4m_1m_2 \int_{\mathbf{R}_+^2} u^{2m_1-1} v^{2m_2-1} P(x_1 \geq u, x_2 \geq v) dudv$$

Now, by the subindependence of complements of coordinate slabs,

$$P(x_1 \geq u, x_2 \geq v) \leq P(x_1 \geq u) P(x_2 \geq v)$$

and therefore,

$$4m_1m_2 \int_{\mathbf{R}_+^2} u^{2m_1-1} v^{2m_2-1} P(x_1 \geq u, x_2 \geq v) dudv \leq$$

$$\leq \quad 4m_1m_2 \int_{\mathbf{R}_+^2} u^{2m_1-1} v^{2m_2-1} P(x_1 \geq u) P(x_2 \geq v) dudv$$

$$= \quad \left(\int_0^\infty 2m_1 u^{2m_1-1} P(x_1 \geq u) du \right) \left(\int_0^\infty 2m_2 v^{2m_2-1} P(x_2 \geq v) dv \right)$$

Using Fubini again, $\int_0^\infty 2m_1 u^{2m_1-1} P(x_1 \geq u) du = \int_{K\cap\{x_1\geq 0\}} x_1^{2m_1}$, and $\int_0^\infty 2m_2 v^{2m_2-1} P(x_2 \geq v) dv = \int_{K\cap\{x_2\geq 0\}} x_2^{2m_2}$, so the following inequality holds:

$$\int_{K\cap\{x_1\geq 0, x_2\geq 0\}} x_1^{2m_1} x_2^{2m_2} \leq \int_{K\cap\{x_1\geq 0\}} x_1^{2m_1} \int_{K\cap\{x_2\geq 0\}} x_2^{2m_2}$$

This clearly suffices for the proof.

The following Lemma, is based on standard results for log-concave functions, whose origins date back to the works of Schur and Ostrowski.

Lemma 6.2 *If f is a decreasing log-concave function, then for any positive integer m the following relation holds:*

$$\left(\int_0^\infty f(x)dx \right)^{m-1} \int_0^\infty x^{2m} f(x)dx \leq \frac{(2m)!}{2} 3^{m-1} \left(\int_0^\infty x^2 f(x)dx \right)^m$$

Proof: We shall use the two following inequalities which are true for all decreasing log-concave functions, and k positive integer:

$$\frac{\int_t^\infty f(x)dx}{f(t)} \leq \frac{\int_0^\infty f(x)dx}{f(0)} \qquad \text{when } t \geq 0 \tag{6.2}$$

and

$$\left(\int_0^\infty f(x)dx\right)^k \leq k\,(f(0))^{k-1} \int_0^\infty x^{k-1} f(x)dx \tag{6.3}$$

We firstly prove the following inductive relation:

$$\int_0^\infty x^k f(x)dx \leq k\frac{\int_0^\infty f(x)dx}{f(0)} \int_0^\infty x^{k-1} f(x)dx \tag{6.4}$$

Indeed,

$$\begin{aligned}
\int_0^\infty x^k f(x)dx &= \int_0^\infty f(x)\left(\int_0^x kt^{k-1}dt\right)dx \\
&= \int_0^\infty kt^{k-1}\left(\int_t^\infty f(x)dx\right)dt \\
&\leq k\frac{\int_0^\infty f(x)dx}{f(0)} \int_0^\infty t^{k-1} f(t)dt
\end{aligned}$$

and in case that $k \geq 2$ we can apply (6.4) $k-2$ successive times, to get:

$$\begin{aligned}
\int_0^\infty x^k f(x)dx &\leq k(k-1)\cdots 3\left(\frac{\int_0^\infty f(x)dx}{f(0)}\right)^{k-2} \int_0^\infty x^2 f(x)dx \\
&= \frac{k!}{2}\left(\frac{\int_0^\infty f(x)dx}{f(0)}\right)^{k-2} \int_0^\infty x^2 f(x)dx
\end{aligned}$$

So, for $k = 2m$

$$\left(\int_0^\infty f(x)dx\right)^{m-1} \int_0^\infty x^{2m} f(x)dx \leq$$

$$\leq \frac{(2m)!}{2}\frac{\left(\int_0^\infty f(x)dx\right)^{3m-3}}{(f(0))^{2m-2}} \int_0^\infty x^2 f(x)dx \tag{6.5}$$

Now using (6.3) for $k = 3$ and taking the $(m-1)$-th power, we get:

$$\left(\int_0^\infty f(x)dx\right)^{3m-3} \leq (f(0))^{2m-2}\, 3^{m-1} \left(\int_0^\infty x^2 f(x)dx\right)^{m-1} \tag{6.6}$$

which combined with (6.5), gives the desired result.

Corollary 6.2.1 *For all positive integers m,*

$$\int_K x_i^{2m} \leq \frac{(2m)!}{2} 3^{m-1} \left(\int_K x_i^2\right)^m \tag{6.7}$$

Proof: It is easy to check that for all positive integers m,

$$\int_K x_i^{2m} = \lambda^{2m} \frac{\int_0^1 u^{2m}(1-u^p)^{\frac{n-1}{p}} du}{\int_0^1 (1-u^p)^{\frac{n-1}{p}} du} \tag{6.8}$$

and then the relation we want to prove becomes:

$$\lambda^{2m} \frac{\int_0^1 u^{2m}(1-u^p)^{\frac{n-1}{p}} du}{\int_0^1 (1-u^p)^{\frac{n-1}{p}} du} \leq \frac{(2m)!}{2} 3^{m-1} \left(\lambda^2 \frac{\int_0^1 u^2(1-u^p)^{\frac{n-1}{p}} du}{\int_0^1 (1-u^p)^{\frac{n-1}{p}} du}\right)^m$$

$$\Leftrightarrow \left(\int_0^1 (1-u^p)^{\frac{n-1}{p}} du\right)^{m-1} \int_0^1 u^{2m}(1-u^p)^{\frac{n-1}{p}} du \leq$$

$$\leq \frac{(2m)!}{2} 3^{m-1} \left(\int_0^1 u^2(1-u^p)^{\frac{n-1}{p}} du\right)^m$$

Which is true by Lemma 6.2, since the function $f(u) = (1-u^p)^{\frac{n-1}{p}}$ is decreasing and log-concave.

Lemma 6.3 *For all positive numbers r,*

$$P\left(\left|\frac{|x|^2}{n} - \varrho_n^2\right| \geq r\right) \leq \frac{35}{nr^2} \varrho_n^4$$

Proof: We first prove that $\frac{1}{n^2}\int_K |x|^4$ is close to ϱ_n^4, namely that,

$$\varrho_n^4 \leq \frac{1}{n^2}\int_K |x|^4 \leq \left(1+\frac{35}{n}\right)\varrho_n^4 \tag{6.9}$$

The first inequality is obvious by Cauchy-Schwartz.

For the second one, we have:

$$\begin{aligned}
\int_K |x|^4 &= \int_K \left(\sum_{i=1}^n x_i^2\right)^2 \\
&= \sum_1^n \int_K x_i^4 + \sum_{i\neq i}\int_K x_i^2 x_j^2 \\
&\leq n\int_K x_i^4 + \sum_{i\neq i}\int_K x_i^2 \int_K x_j^2
\end{aligned}$$

Using the Corollary of Lemma 6.2, $\int_K x_i^4 \leq 36\left(\int_K x_i^2\right)^2 = 36\varrho_n^4$. Thus,

$$\begin{aligned}
\int_K |x|^4 &\leq 36n\varrho_n^4 + n(n-1)\varrho_n^4 \\
&= n^2\left(1+\frac{35}{n}\right)\varrho_n^4
\end{aligned}$$

From this we can conclude that the integral $\int_K\left(\frac{|x|^2}{n}-\varrho_n^2\right)^2$ is small, and therefore that $\frac{|x|^2}{n}$ is close to ϱ_n^2. Indeed,

$$\begin{aligned}
0 \leq \int_K\left(\frac{|x|^2}{n}-\varrho_n^2\right)^2 &= \frac{1}{n^2}\int_K |x|^4 - \frac{2}{n}\varrho_n^2\int_K |x|^2 + \varrho_n^4 \\
&= \frac{1}{n^2}\int_K |x|^4 - \frac{2}{n}\varrho_n^2 n\varrho_n^2 + \varrho_n^4 \\
&= \frac{1}{n^2}\int_K |x|^4 - \varrho_n^4 \\
&\leq \frac{35}{n}\varrho_n^4
\end{aligned}$$

The last inequality is true by (6.9).

Finally by Chebychev's inequality we have:

$$
\begin{aligned}
P\left(\left|\frac{|x|^2}{n} - \varrho_n^2\right| \geq r\right) r^2 &= P\left(\left(\frac{|x|^2}{n} - \varrho_n^2\right)^2 \geq r^2\right) r^2 \\
&\leq \int_K \left(\frac{|x|^2}{n} - \varrho_n^2\right)^2 \\
&\leq \frac{35}{n}\varrho_n^4
\end{aligned}
$$

Lemma 6.3 deals comfortably with the possibility that $|x|$ might be too large, but for technical reasons we need a stronger estimate from below. Fortunately, this can be deduced from Lemma 6.3 using the logarithmic concavity of the function

$$s \mapsto Vol_n(K \cap B(s\sqrt{n}))$$

where $B(s\sqrt{n})$ is the Euclidean ball of radius $s\sqrt{n}$.

Lemma 6.4 *For* $r \geq \frac{1}{n^{1/6}}$*,*

$$P\left(\frac{|x|}{\sqrt{n}} \leq \varrho_n - r\right) \leq \left(\frac{2}{3}\right)^{n^{1/6}r}$$

Proof: Let $\widehat{\varrho}_n$, be such that $P\left(\frac{|x|}{\sqrt{n}} \leq \widehat{\varrho}_n\right) = 1/2$. It is easy to see by Lemma 6.3 that $|\varrho_n - \widehat{\varrho}_n| \leq \frac{1}{2n^{1/6}}$. Combining this with Lemma 6.3 again, we get that when n is large enough,

$$P\left(\frac{|x|}{\sqrt{n}} \leq \widehat{\varrho}_n + \frac{1}{n^{1/6}}\right) \geq \frac{3}{4} \tag{6.10}$$

So, we have a point $\widehat{\varrho}_n$, at which the log-concave function $f(s) = P\left(\frac{|x|}{\sqrt{n}} \leq s\right)$ takes the value 1/2, and a point just a bit further on, where it takes a value close to 1. So, using the log-concavity of the function, we have an estimate for its value at points before $\widehat{\varrho}_n$. Indeed, take $s \le \widehat{\varrho}_n$, and $\lambda = \frac{n^{1/6}(\widehat{\varrho}_n - s)}{1+n^{1/6}(\widehat{\varrho}_n - s)}$. Then,

$$\widehat{\varrho}_n = \lambda \left(\widehat{\varrho}_n + \frac{1}{n^{1/6}} \right) + (1-\lambda)s$$

and thus by log-concavity

$$f(\widehat{\varrho}_n) \ge f^{\lambda} \left(\widehat{\varrho}_n + \frac{1}{n^{1/6}} \right) \cdot f^{1-\lambda}(s)$$

But since $f(\widehat{\varrho}_n) = 1/2$ and $f\left(\widehat{\varrho}_n + \frac{1}{n^{1/6}}\right) \ge 3/4$ the above relation implies that

$$f(s) \le \left(\frac{2}{3}\right)^{1/(1-\lambda)}$$

and hence that

$$P\left(\frac{|x|}{\sqrt{n}} \le s\right) \le \left(\frac{2}{3}\right)^{1+n^{1/6}(\widehat{\varrho}_n - s)}$$

Now we need only notice that for $s \le \varrho_n - \frac{1}{n^{1/6}} \le \widehat{\varrho}_n$,

$$P\left(\frac{|x|}{\sqrt{n}} \le s\right) \le \left(\frac{2}{3}\right)^{1+n^{1/6}(\widehat{\varrho}_n - s)} \le \left(\frac{2}{3}\right)^{n^{1/6}(\varrho_n - s)}$$

and then put $r = \varrho_n - s$ to get the required result.

Theorem 6.1 *If g_θ is the density of the marginal in direction θ, of the ℓ_p^n-ball, then*

$$\int_{S^{n-1}} g_\theta(t) d\sigma(\theta) \quad \longrightarrow \quad \frac{1}{\varrho\sqrt{2\pi}} \exp\left(-\frac{t^2}{2\varrho^2}\right) \qquad \textit{as } n \to \infty$$

for each t, uniformly in p.

Proof: By (6.1), it is sufficient to prove that

$$\int_K \frac{1}{\sqrt{2\pi}} \frac{\sqrt{n}}{|x|} \exp\left(-\frac{t^2}{2}\frac{n}{|x|^2}\right) \quad \longrightarrow \quad \frac{1}{\varrho\sqrt{2\pi}} \exp\left(-\frac{t^2}{2\varrho^2}\right)$$

for each t, uniformly in p. For this, we shall divide the set K into two subsets and use appropriate techniques in each one. These are: $K_1 = K \cap \left\{\left|\frac{|x|}{\sqrt{n}} - \varrho_n\right| \le \frac{\log n}{n^{1/6}}\right\}$ and $K_2 = K \cap \left\{\left|\frac{|x|}{\sqrt{n}} - \varrho_n\right| \ge \frac{\log n}{n^{1/6}}\right\}$.

We shall show that the mass of K is concentrated in the first set, where the integrated function is pretty smooth. By applying a Lipschitz estimate, we shall see that the limit of the integral there, is the required Gaussian. Although the integrand is not particularly well-behaved on K_2, the measure of K_2 is small, so the integral will tend to zero.

Our aim is to prove the following two statements:

$$\lim_{n\to\infty} \int_{K_1} \frac{1}{\sqrt{2\pi}} \frac{\sqrt{n}}{|x|} \exp\left(-\frac{t^2}{2}\frac{n}{|x|^2}\right) = \frac{1}{\varrho\sqrt{2\pi}} \exp\left(-\frac{t^2}{2\varrho^2}\right) \tag{6.11}$$

$$\lim_{n\to\infty} \int_{K_2} \frac{1}{\sqrt{2\pi}} \frac{\sqrt{n}}{|x|} \exp\left(-\frac{t^2}{2}\frac{n}{|x|^2}\right) = 0 \tag{6.12}$$

- For the first one, as was already mentioned, we shall use the fact that the integrand is smooth in this set.

  Let $F(y) = y^{-1}\exp\left(-\frac{t^2}{2y^2}\right)$. The derivative of F, is bounded by t^{-2}. Therefore, if t is large, say $t \ge \frac{\varrho_n}{3}$, this gives an upper bound for the derivative of order ϱ_n^{-2}. When t is small, the function F can have large derivative, but only where y is small. More precisely, for $y \ge 2t$, the derivative is decreasing, so, one can check that in case that $t \le \frac{\varrho_n}{3}$, the derivative in a range near ϱ_n, has again a bound of order ϱ_n^{-2}. Thus, for $y \ge \varrho_n - \frac{\log n}{n^{1/6}}$, we have a Lipschitz property:

$$|F(y) - F(\varrho_n)| \le c|y - \varrho_n|$$

where c is of order ϱ_n^{-2}. This applied in the integrals, gives:

$$\left|\int_{K_1}\left(\frac{1}{\sqrt{2\pi}}\frac{\sqrt{n}}{|x|}\exp\left(-\frac{t^2}{2}\frac{n}{|x|^2}\right)-\frac{1}{\varrho_n\sqrt{2\pi}}\exp\left(-\frac{t^2}{2\varrho_n^2}\right)\right)\right| \leq$$

$$\begin{aligned}
&\leq \int_{K_1}\left|\frac{1}{\sqrt{2\pi}}\frac{\sqrt{n}}{|x|}\exp\left(-\frac{t^2}{2}\frac{n}{|x|^2}\right)-\frac{1}{\varrho_n\sqrt{2\pi}}\exp\left(-\frac{t^2}{2\varrho_n^2}\right)\right| \\
&\leq \int_{K_1}\frac{c}{\sqrt{2\pi}}\left|\frac{|x|}{\sqrt{n}}-\varrho_n\right| \\
&\leq \frac{\log n}{n^{1/6}}\cdot\frac{c}{\sqrt{2\pi}}Vol_n(K_1) \\
&\longrightarrow 0
\end{aligned}$$

Since $\varrho_n \longrightarrow \varrho$ and $Vol\,(K_1) \longrightarrow 1$ (by Lemma 6.3), we have that

$$\int_{K_1}\frac{1}{\varrho_n\sqrt{2\pi}}\exp\left(-\frac{t^2}{2\varrho_n^2}\right) \quad\longrightarrow\quad \frac{1}{\varrho\sqrt{2\pi}}\exp\left(-\frac{t^2}{2\varrho^2}\right)$$

which completes the proof of (6.11).

- To prove (6.12), we notice that it is enough to prove:

$$\lim_{n\to\infty}\int_{K_2}\frac{\sqrt{n}}{|x|}=0 \tag{6.13}$$

We divide K_2 into three subsets. The first one has a small radius. This is $K_{2,1}=K\cap\left\{\frac{|x|}{\sqrt{n}}\leq\frac{l}{\sqrt{n}}\right\}$ where l is chosen such that $v_n l^n\sqrt{n}=1$, where v_n is the volume of the unit Euclidean ball. So, l is like a constant times $\sqrt{n}$. The other two subsets of K_2 are: $K_{2,2}=K\cap\left\{\frac{l}{\sqrt{n}}\leq\frac{|x|}{\sqrt{n}}\leq\varrho_n-\frac{\log n}{n^{1/6}}\right\}$ and $K_{2,3}=K\cap\left\{\frac{|x|}{\sqrt{n}}\geq\varrho_n+\frac{\log n}{n^{1/6}}\right\}$

For the first one we have:

$$\lim_{n\to\infty}\int_{K_{2,1}}\frac{\sqrt{n}}{|x|} \;\leq\; \lim_{n\to\infty}\int_{B(l)}\frac{\sqrt{n}}{|x|}$$

$$
\begin{aligned}
&= \lim_{n\to\infty} v_n n\sqrt{n} \int_{S^{n-1}} \int_0^l \frac{1}{u} \cdot u^{n-1} du \, d\sigma(\theta) \\
&= \lim_{n\to\infty} v_n l^{n-1} \sqrt{n} \frac{n}{n-1} \\
&= 0
\end{aligned}
$$

For the second one, we shall use Lemma 6.4. We have:

$$
\begin{aligned}
\lim_{n\to\infty} \int_{K_{2,2}} \frac{\sqrt{n}}{|x|} &\leq \lim_{n\to\infty} \frac{\sqrt{n}}{l} Vol_n(K_{2,2}) \\
&\leq \lim_{n\to\infty} \frac{\sqrt{n}}{l} P\left(\frac{|x|}{\sqrt{n}} \leq \varrho - \frac{\log n}{n^{1/6}} \right) \\
&\leq \lim_{n\to\infty} \frac{\sqrt{n}}{l} \left(\frac{2}{3} \right)^{\log n} \\
&= 0
\end{aligned}
$$

Finally, for the last one, we shall use Lemma 6.3:

$$
\begin{aligned}
\lim_{n\to\infty} \int_{K_{2,3}} \frac{\sqrt{n}}{|x|} &\leq \lim_{n\to\infty} \frac{1}{\varrho_n + \log n / n^{1/6}} Vol_n(K_{2,3}) \\
&\leq \lim_{n\to\infty} \frac{1}{\varrho_n + \log n / n^{1/6}} \cdot \frac{35 \varrho_n^4}{n^{1/3} \log^4 n} \\
&= 0
\end{aligned}
$$

(The choice of $\log n$ in the above argument is not crucial: we need a function of n which tends to infinity, to handle the case of $K_{2,2}$, but more slowly than $n^{1/6}$, to handle the case of K_1.)

Remark

1. In the statement of Theorem 6.1 we write that the convergence is uniformly in "p". As the proof stands, this would be clear, if all the ϱ_n's where uniformly bounded.

But this is not difficult to see: From the discussion in the beginning of this Chapter, we mentioned that there is a constant c_p depending only on p, such that as $n \longrightarrow \infty$,

$$\varrho_n \longrightarrow c_p \int_0^\infty y^2 \exp\left(-\frac{y^p}{p}\right) dy$$

As it can be observed by Stirling's formula, these c_p's are bounded in p. And since the integral is also bounded in p, we have what we want.

Appendix

Using the notation introduced in Chapter 3, we prove in the Lemma below, that the function $1-\frac{Y_n(t)}{Y_n(1)}$ is very much like the function $(1-t^p)^{\frac{n}{p}}$, a property used in the proof of Theorem 3.1.

Its proof uses standard inequalities for log-concave functions.

Lemma A.1 *For $p \geq 1$ and $0 \leq t \leq 1$,*

$$(1-t^p)^{\frac{n}{p}} \geq 1-\frac{Y_n(t)}{Y_n(1)} \geq \frac{1}{2p}(1-t^p)^{\frac{n}{p}}$$

Proof:

We shall use the form: $1-\frac{Y_n(t)}{Y_n(1)} = \frac{\int_t^1 (1-u^p)^{\frac{n-1}{p}}}{\int_0^1 (1-u^p)^{\frac{n-1}{p}}}$

For the first inequality, we have:

$$\begin{aligned} \int_t^1 (1-u^p)^{\frac{n-1}{p}} du &= (1-t^p)^{\frac{n-1}{p}} \int_t^1 \left(\frac{1-u^p}{1-t^p}\right)^{\frac{n-1}{p}} du \\ &= (1-t^p)^{\frac{n-1}{p}} \int_t^1 \left(1-\frac{u^p-t^p}{1-t^p}\right)^{\frac{n-1}{p}} du \end{aligned}$$

Now substituting $x^p = \frac{u^p-t^p}{1-t^p}$, we get:

$$\int_t^1 (1-u^p)^{\frac{n-1}{p}} du = (1-t^p)^{\frac{n-1}{p}} \int_0^1 (1-t^p)\frac{x^{p-1}}{u^{p-1}}(1-x^p)^{\frac{n-1}{p}} dx$$

$$
\begin{aligned}
&\leq (1-t^p)^{\frac{n-1}{p}+1}\int_0^1 (1-x^p)^{\frac{n-1}{p}}dx \\
&\leq (1-t^p)^{\frac{n}{p}}\int_0^1 (1-x^p)^{\frac{n-1}{p}}dx
\end{aligned}
$$

Which completes the proof of the first inequality.

For the second one, we shall use the following inequality which holds for a decreasing log-concave function f, and for $p \geq 1$:

$$
p(f(t))^{p-1}\int_t^\infty x^{p-1}f(x)dx \leq \Gamma(p+1)\left(\int_t^\infty f(x)dx\right)^p \tag{A.1}
$$

Applying this for $f(x)=(1-x^p)^{\frac{n-1}{p}}$, we get that:

$$
\begin{aligned}
\Gamma(p+1)\left(\int_t^1 (1-u^p)^{\frac{n-1}{p}}du\right)^p &\geq (1-t^p)^{\frac{n-1}{p}(p-1)}\int_t^1 pu^{p-1}(1-u^p)^{\frac{n-1}{p}}du \\
&\geq (1-t^p)^{\frac{n-1}{p}(p-1)}\int_{t^p}^1 (1-x)^{\frac{n-1}{p}}dx \\
&= (1-t^p)^{\frac{n-1}{p}(p-1)}\frac{p}{n-1+p}(1-t^p)^{\frac{n-1}{p}+1} \\
&= (1-t^p)^n\frac{p}{n-1+p}
\end{aligned}
$$

But then,

$$
\begin{aligned}
\int_t^1 (1-u^p)^{\frac{n-1}{p}}du &\geq (1-t^p)^{n/p}\left(\frac{p}{n-1+p}\right)^{1/p}(\Gamma(p+1))^{-1/p} \\
&\geq (1-t^p)^{n/p}\left(\frac{p}{n-1+p}\right)^{1/p}\frac{1}{p} \qquad \text{(A.2)}
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
\int_0^1 (1-u^p)^{\frac{n-1}{p}}du &= \frac{1}{p}\cdot \mathrm{B}\left(\frac{1}{p},1+\frac{n-1}{p}\right) \\
&= \frac{\Gamma\left(1+\frac{1}{p}\right)\Gamma\left(1+\frac{n-1}{p}\right)}{\Gamma\left(1+\frac{n}{p}\right)} \\
&\leq \frac{\Gamma\left(1+\frac{n-1}{p}\right)}{\Gamma\left(1+\frac{n}{p}\right)} \qquad \text{(A.3)}
\end{aligned}
$$

Applying Lemma A.2 with $x = \frac{n}{p}$ and $\alpha = \frac{1}{p}$ we get

$$\frac{\Gamma\left(1 + \frac{n-1}{p}\right)}{\Gamma\left(1 + \frac{n}{p}\right)} \leq \left(\frac{n}{p}\right)^{-1/p}$$

Thus, by (A.3)

$$\int_0^1 (1-u^p)^{\frac{n-1}{p}} du \leq \left(\frac{n}{p}\right)^{-1/p}$$

This, combined with (A.2), gives:

$$\begin{aligned}
\frac{\int_t^1 (1-u^p)^{\frac{n-1}{p}} du}{\int_0^1 (1-u^p)^{\frac{n-1}{p}} du} &\geq (1-t^p)^{n/p} \left(\frac{n}{p}\right)^{1/p} \left(\frac{p}{n-1+p}\right)^{1/p} \frac{1}{p} \\
&= (1-t^p)^{n/p} \left(\frac{n}{n-1+p}\right)^{1/p} \frac{1}{p} \\
&\geq \frac{1}{2p}(1-t^p)^{n/p}
\end{aligned}$$

which is what we want.

In the next Lemma we give a property for Euler's Gamma function Γ, used to prove an accurate upper bound for the integral $\int_0^1 (1-u^p)^{\frac{n-1}{p}} du$

Lemma A.2 *For all $x \geq 0$ and $0 \leq \alpha \leq 1$, Euler's Gamma function Γ, satisfies:*

$$\frac{\Gamma(1+x-\alpha)}{\Gamma(1+x)} \leq x^{-\alpha}$$

Proof : We shall use Gauss' formula for the Gamma function:

$$\Gamma(x) = \lim_{n\to\infty} \frac{n! \cdot n^x}{x(x+1)\cdots(x+n)}$$

Then,

$$\frac{\Gamma(1+x-\alpha)}{\Gamma(1+x)} = \lim_{n\to\infty} \left[n^\alpha \left(1 - \frac{\alpha}{x+1}\right) \cdots \left(1 - \frac{\alpha}{x+n}\right)\right]^{-1}$$

But since

$$\lim_{n\to\infty}\left[n^{\alpha}\left(1-\frac{\alpha}{x+1}\right)\cdots\left(1-\frac{\alpha}{x+n}\right)\right]^{-1}=$$

$$=\lim_{n\to\infty}\left[(x+n)^{\alpha}\left(1-\frac{\alpha}{x+1}\right)\cdots\left(1-\frac{\alpha}{x+n}\right)\right]^{-1}$$

it is enough to show that

$$\left(1-\frac{\alpha}{x+1}\right)\cdots\left(1-\frac{\alpha}{x+n}\right)\geq\left(\frac{x}{x+n}\right)^{\alpha} \tag{A.4}$$

Indeed, using the fact that $1-\alpha s\geq(1-s)^{\alpha}$ for all $0\leq s\leq 1$ and $0\leq\alpha\leq 1$, we get:

$$\begin{aligned}\left(1-\frac{\alpha}{x+1}\right)\cdots\left(1-\frac{\alpha}{x+n}\right) &\geq \left[\left(1-\frac{1}{x+1}\right)\cdots\left(1-\frac{1}{x+n}\right)\right]^{\alpha}\\ &= \left(\frac{x}{x+1}\cdots\frac{x+n-1}{x+n}\right)^{\alpha}\\ &= \left(\frac{x}{x+n}\right)^{\alpha}\end{aligned}$$

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