Geometriae Dedicata **84:** 63–79, 2001.

John's Theorem for an Arbitrary Pair of Convex Bodies

A. GIANNOPOULOS[1], I. PERISSINAKI[1] and A. TSOLOMITIS[2]
[1]*Department of Mathematics, University of Crete, Iraklion, Greece.*
e-mail: {apostolo, irinip}@math.uch.gr
[2]*Department of Mathematics, University of the Aegean, Samos, Greece.*
e-mail: atsol@iris.math.aegean.gr

(Received: 6 May 1999)

Communicated by R. Schneider

Abstract. We provide a generalization of John's representation of the identity for the maximal volume position of L inside K, where K and L are arbitrary smooth convex bodies in $\mathbb{R}^n$. From this representation we obtain Banach–Mazur distance and volume ratio estimates.

Mathematics Subject Classifications (2000). Primary: 52A21; Secondary: 52A40, 52A38, 46B07.

Key words. John's theorem, Banach–Mazur distance, volume ratio.

1. Introduction

The definition of the Banach–Mazur distance between symmetric convex bodies can be extended to the nonsymmetric case as follows [Gr]: Let K and L be two convex bodies in $\mathbb{R}^n$. Their geometric distance is defined by

$$\tilde{d}(K, L) = \inf\{ab \colon (1/b)L \subseteq K \subseteq aL\}. \tag{1}$$

If $z_1, z_2 \in \mathbb{R}^n$, we consider the translates $K - z_1$ and $L - z_2$ of K and L, and their distance with respect to z_1, z_2,

$$d_{z_1,z_2}(K, L) = \inf\{\tilde{d}(T(K - z_1), L - z_2)\}, \tag{2}$$

where the inf is taken over all invertible linear transformations T of $\mathbb{R}^n$. Finally, we let z_1, z_2 vary and define

$$d(K, L) = \inf\{d_{z_1,z_2}(K, L) \colon z_1, z_2 \in \mathbb{R}^n\}. \tag{3}$$

John's theorem [J] provides a first estimate for $d(K, L)$. If K is any convex body in $\mathbb{R}^n$ and E is its maximal or minimal volume ellipsoid, then $d_{z,z}(K, E) \leqslant n$, where z is the center of E. Actually, the distance between the simplex and the ball is equal to n, and the simplex is the only body with this property [P]. It follows that the distance between any two convex bodies is at most n^2. Rudelson [R] has recently proved that $d(K, L) \leqslant cn^{4/3} \log^{\beta} n$ for some absolute constants $c, \beta > 0$ (see also recent work of Litvak and Tomczak–Jaegermann [LTJ]). A well-known theorem of Gluskin [Gl] shows that $d(K, L)$ can be of the order of n even for symmetric bodies K and L.

In this paper we study the *maximal volume position* of a body L inside K: we say that L *is of maximal volume in* K if $L \subseteq K$ and, for every $w \in \mathbb{R}^n$ and every volume preserving linear transformation T of $\mathbb{R}^n$, the affine image $w + T(L)$ of L is not contained in the interior of K. A simple compactness argument shows that for every pair of convex bodies K and L there exists an affine image $\tilde{L}$ of L which is of maximal volume in K. Note that the maximal volume position of L inside K is not unique, as it can be seen by the example of a simplex inside the cube.

Our main result is the following theorem:

THEOREM. *Let L be of maximal volume in K. If $z \in \mathrm{int}(L)$, we can find contact points $v_1, \ldots, v_m$ of $K - z$ and $L - z$, contact points $u_1, \ldots, u_m$ of the polar bodies $(K - z)^\circ$ and $(L - z)^\circ$, and positive reals $\lambda_1, \ldots, \lambda_m$, such that: $\sum \lambda_j u_j = o$, $\langle u_j, v_j \rangle = 1$, and*

$$\mathrm{Id} = \sum_{j=1}^{m} \lambda_j u_j \otimes v_j. \tag{4}$$

We shall prove the above fact under the assumption that both K and L are smooth enough. The theorem may be viewed as a generalization of John's representation of the identity even in the case where L is the Euclidean unit ball. This generalization was observed by V. D. Milman in the case where K and L are o-symmetric and $z = o$ (see [TJ], Theorem 14.5).

Using the theorem, we give a direct proof of the fact that $d(K, L) \leqslant n$ when both K and L are symmetric, and we obtain the estimate $d(K, L) \leqslant 2n - 1$ when L is symmetric and K is any convex body (this was recently proved by Lassak [L]).

Note that the theorem holds true for any choice of $z \in \mathrm{int}(L)$. In Section 3 we prove an extension to the case $z \in \mathrm{bd}(L)$. Also, assuming that L is a polytope and K has C^2 boundary with strictly positive curvature, we show that the center z may be chosen so that $\sum \lambda_j u_j = o = \sum \lambda_j v_j$.

Using the maximal volume position of L inside K, one can naturally extend the notion of *volume ratio* to an arbitrary pair of convex bodies. We define

$$\mathrm{vr}(K, L) = \left(\frac{|K|}{|\tilde{L}|} \right)^{\frac{1}{n}}, \tag{5}$$

where $\tilde{L}$ is an affine image of L which is of maximal volume in K (by $|\cdot|$ we denote n-dimensional volume). In Section 4, we prove the following general estimate:

THEOREM. *Let K and L be two convex bodies in $\mathbb{R}^n$. Then,*

$$\mathrm{vr}(K, L) \leqslant n. \tag{6}$$

The same estimate can be given through known results on $\mathrm{vr}(K, D_n)$ and $\mathrm{vr}(D_n, K)$, where D_n is the Euclidean unit ball. Ball [Ba] proved that $\mathrm{vr}(K, D_n)$ is maximal when K is the simplex S_n, and noticed that from the reverse Brascamp–Lieb inequality (which was later proved in [Bar]) it would follow that $\mathrm{vr}(D_n, K)$ is also maximal for S_n. It follows that

$$\mathrm{vr}(K, L) \leqslant \mathrm{vr}(K, D_n)\mathrm{vr}(D_n, L) \leqslant \mathrm{vr}(S_n, D_n)\mathrm{vr}(D_n, S_n) = n.$$

However, our proof is direct and might lead to a better estimate; it might be true that $\mathrm{vr}(K, L)$ is always bounded by $c\sqrt{n}$.

2. The Main Theorem and Distance Estimates

We assume that $\mathbb{R}^n$ is equipped with a Euclidean structure $\langle \cdot, \cdot \rangle$, and denote the corresponding Euclidean norm by $| \cdot |$. We write D_n for the Euclidean unit ball, and S^{n-1} for the unit sphere.

If W is a convex body in $\mathbb{R}^n$ and $z \in \mathrm{int}(W)$, we define the radial function $\rho_W(z, \cdot)$ of W with respect to z by

$$\rho_W(z, \theta) = \max\{\lambda > 0 : z + \lambda\theta \in W\} \tag{1}$$

for $\theta \in S^{n-1}$, and extend this definition to $\mathbb{R}^n \backslash \{z\}$ by

$$\rho_W(z, x) = \frac{1}{t}\rho_W(z, \theta), \tag{2}$$

where $x = z + t\theta$, $t > 0$ and $\theta \in S^{n-1}$. If $\theta \in S^{n-1}$, we will write $\rho_W(z, \theta)$ instead of $\rho_W(z, z + \theta)$ (this will cause no confusion).

The polar body W^z of W with respect to $z \in \mathrm{int}(W)$ is the body

$$W^z = (W - z)^\circ = \{y \in \mathbb{R}^n : \langle y, x - z\rangle \leqslant 1 \text{ for all } x \in W\} \tag{3}$$

(some authors write W^z for $(W - z)^\circ + z$).

Let o denote the origin. Since $\rho_W(z, x) = \rho_{W-z}(o, x - z)$, the support function h_{W^z} of W^z satisfies

$$h_{W^z}(x - z) = \frac{1}{\rho_W(z, x)} \tag{4}$$

for all $x \in \mathbb{R}^n \backslash \{z\}$. Note that the definition of the polar set W^z makes sense for $z \in \mathrm{bd}(W)$, but then W^z may be unbounded in some directions.

Recall that, if $o \in \mathrm{int}(W)$, W is strictly convex and h_W is continuously differentiable, then $\nabla h_W(\theta)$ is the unique point on the boundary of W at which the outer unit normal to W is θ, and $\nabla h_W(\lambda\theta) = \nabla h_W(\theta)$ for all $\lambda > 0$.

With these definitions, we have the following description of the maximal volume position of L in K:

LEMMA 2.1. *Let K and L be two convex bodies in $\mathbb{R}^n$, with $L \subseteq K$. Then, L is of maximal volume in K if and only if, for every $z \in L$, for every $w \in \mathbb{R}^n$ and every volume preserving T, there exists $\theta \in S^{n-1}$ such that*

$$\rho_K\Big(z, z + w + T(\rho_L(z, \theta)\theta)\Big) \leqslant 1. \qquad \square \tag{5}$$

We assume that K is *smooth enough*: we ask that it is strictly convex and its support function h_K is twice continuously differentiable on $\mathbb{R}^n \setminus \{0\}$. Under this assumption, we have that h_{K^z} is twice continuously differentiable on $\mathbb{R}^n \setminus \{0\}$ for every $z \in \mathrm{int}(K)$.

LEMMA 2.2. *Let L be of maximal volume in K, and $z \in L \cap \mathrm{int}(K)$. Then, for every $w \in \mathbb{R}^n$ and every $S \in L(\mathbb{R}^n, \mathbb{R}^n)$ we can find $\theta \in S^{n-1}$ such that $\rho_L(z, \theta) = \rho_K(z, \theta)$ and*

$$h_{K^z}\Big(w + \rho_K(z, \theta)S(\theta)\Big) \geqslant \frac{\mathrm{tr} S}{n}. \tag{6}$$

Proof. We follow the argument of [GM]. Let $w \in \mathbb{R}^n$ and $S \in L(\mathbb{R}^n, \mathbb{R}^n)$. If $\varepsilon > 0$ is small enough, then $T_\varepsilon = (I + \varepsilon S)/[\det(I + \varepsilon S)]^{1/n}$ is volume preserving, hence, using (4) and Lemma 2.1 for T_ε and εw, we find $\theta_\varepsilon \in S^{n-1}$ such that

$$h_{K^z}\Big(\varepsilon w + T_\varepsilon(\rho_L(z, \theta_\varepsilon)\theta_\varepsilon)\Big) \geqslant 1. \tag{7}$$

Since

$$[\det(I + \varepsilon S)]^{1/n} = 1 + \varepsilon \frac{\mathrm{tr} S}{n} + \mathrm{O}(\varepsilon^2),$$

we get

$$h_{K^z}\Big(\rho_L(z, \theta_\varepsilon)\theta_\varepsilon + \varepsilon w + \varepsilon \rho_L(z, \theta_\varepsilon)S(\theta_\varepsilon)\Big) \geqslant 1 + \varepsilon \frac{\mathrm{tr} S}{n} + \mathrm{O}(\varepsilon^2). \tag{8}$$

Since $L \subseteq K$, we have $h_{K^z}(\rho_L(z, \theta_\varepsilon)\theta_\varepsilon) = \rho_L(z, \theta_\varepsilon)/\rho_K(z, \theta_\varepsilon) \leqslant 1$, and the subadditivity of h_{K^z} gives

$$h_{K^z}\Big(w + \rho_L(z, \theta_\varepsilon)S(\theta_\varepsilon)\Big) \geqslant \frac{\mathrm{tr} S}{n} + \mathrm{O}(\varepsilon). \tag{9}$$

By compactness, we can find $\varepsilon_m \to 0$ and $\theta \in S^{n-1}$ such that $\theta_{\varepsilon_m} \to \theta$. Then, taking limits in (9), we get

$$h_{K^z}\Big(w + \rho_L(z, \theta)S(\theta)\Big) \geqslant \frac{\mathrm{tr} S}{n}, \tag{10}$$

and taking limits in (7) we see that $h_{K^z}(\rho_L(z, \theta)\theta) \geqslant 1$, which forces $\rho_L(z, \theta) = \rho_K(z, \theta)$. $\square$

Making one more step, we obtain the following condition:

LEMMA 2.3. *Let L be of maximal volume in K, and $z \in L \cap \mathrm{int}(K)$. Then, for every $w \in \mathbb{R}^n$ and every $T \in L(\mathbb{R}^n, \mathbb{R}^n)$ we can find $\theta \in S^{n-1}$ such that $\rho_L(z, \theta) = \rho_K(z, \theta)$ and*

$$\langle \nabla h_{K^z}(\theta), w + \rho_K(z, \theta) T(\theta) \rangle \geqslant \frac{\mathrm{tr} T}{n}. \tag{11}$$

Proof. Let $T \in L(\mathbb{R}^n, \mathbb{R}^n)$, and define $S_\varepsilon = I + \varepsilon T$, $\varepsilon > 0$. By Lemma 2.2, we can find $\theta_\varepsilon \in S^{n-1}$ such that $\rho_K(z, \theta_\varepsilon) = \rho_L(z, \theta_\varepsilon)$ and

$$\begin{aligned} & h_{K^z}(\varepsilon w + \rho_K(z, \theta_\varepsilon)\theta_\varepsilon + \varepsilon \rho_K(z, \theta_\varepsilon) T(\theta_\varepsilon)) \\ & \quad \geqslant \frac{\mathrm{tr}(I + \varepsilon T)}{n} = 1 + \varepsilon \frac{\mathrm{tr} T}{n}. \end{aligned} \tag{12}$$

The left-hand side is equal to

$$\begin{aligned} & h_{K^z}(\rho_K(z, \theta_\varepsilon)\theta_\varepsilon) + \varepsilon \langle \nabla h_{K^z}(\theta_\varepsilon), w + \rho_K(z, \theta_\varepsilon) T(\theta_\varepsilon) \rangle + \mathrm{O}(\varepsilon^2) \\ & \quad = 1 + \varepsilon \langle \nabla h_{K^z}(\theta_\varepsilon), w + \rho_K(z, \theta_\varepsilon) T(\theta_\varepsilon) \rangle + \mathrm{O}(\varepsilon^2). \end{aligned} \tag{13}$$

Therefore,

$$\langle \nabla h_{K^z}(\theta_\varepsilon), w + \rho_K(z, \theta_\varepsilon) T(\theta_\varepsilon) \rangle \geqslant \frac{\mathrm{tr} T}{n} + \mathrm{O}(\varepsilon). \tag{14}$$

Choosing again $\varepsilon_m \to 0$ such that $\theta_{\varepsilon_m} \to \theta \in S^{n-1}$, we see that $\rho_K(z, \theta) = \rho_L(z, \theta)$ and θ satisfies (11). □

Lemma 2.3 and a separation argument give us a generalization of John's representation of the identity:

THEOREM 2.4. *Let K be smooth enough, L be of maximal volume in K, and $z \in L \cap \mathrm{int}(K)$. There exist $m \leqslant n^2 + n + 1$ vectors $\theta_1, \ldots, \theta_m \in S^{n-1}$ such that $\rho_K(z, \theta_j) = \rho_L(z, \theta_j)$ and $\lambda_1, \ldots, \lambda_m > 0$, such that:*

(i) $\sum \lambda_j \nabla h_{K^z}(\theta_j) = o$,
(ii) $\mathrm{Id} = \sum \lambda_j [(\nabla h_{K^z}(\theta_j)) \otimes (\rho_K(z, \theta_j)\theta_j)]$.

Proof. We identify the affine transformations of $\mathbb{R}^n$ with points in $\mathbb{R}^{n^2+n}$, and consider the set

$$\mathcal{C} = \mathrm{co}\Big\{ [\nabla h_{K^z}(\theta) \otimes \rho_K(z, \theta)\theta] + \nabla h_{K^z}(\theta) \colon \theta \in S^{n-1}, \rho_K(z, \theta) = \rho_L(z, \theta) \Big\}. \tag{15}$$

Then, $\mathcal{C}$ is a compact convex set with the Euclidean metric, and we claim that $\mathrm{Id}/n \in \mathcal{C}$. If not, there exist $w \in \mathbb{R}^n$ and $T \in L(\mathbb{R}^n, \mathbb{R}^n)$ such that

$$\langle \mathrm{Id}/n, T + w \rangle > \Big\langle [\nabla h_{K^z}(\theta) \otimes \rho_K(z, \theta)\theta] + \nabla h_{K^z}(\theta), T + w \Big\rangle, \tag{16}$$

whenever $\rho_K(z,\theta) = \rho_L(z,\theta)$. But, (16) is equivalent to

$$\frac{\mathrm{tr}T}{n} > \langle \nabla h_{K^z}(\theta), w + \rho_K(z,\theta)T(\theta)\rangle, \tag{17}$$

and this contradicts Lemma 2.3.

Carathéodory's theorem shows that we can find $m \leqslant n^2 + n + 1$ and positive reals $\lambda_1, \ldots, \lambda_m$ such that

$$\mathrm{Id} = \sum_{j=1}^{m} \lambda_j \Big([\nabla h_{K^z}(\theta_j) \otimes \rho_K(z,\theta_j)\theta_j] + \nabla h_{K^z}(\theta_j) \Big), \tag{18}$$

for $\theta_1, \ldots, \theta_m \in S^{n-1}$ with $\rho_K(z,\theta_j) = \rho_L(z,\theta_j)$. This completes the proof. □

Remark. Assume that L is also smooth enough. Let $\theta \in S^{n-1}$ be such that $\rho_K(z,\theta) = \rho_L(z,\theta)$. Observe that

$$\langle \nabla h_{K^z}(\theta), \rho_K(z,\theta)\theta \rangle = \rho_K(z,\theta) h_{K^z}(\theta) = 1. \tag{19}$$

Also, $x = \nabla h_{L^z}(\theta)$ is the unique point of L^z for which $\langle x, \theta \rangle = h_{L^z}(\theta) = h_{K^z}(\theta)$. Since $\langle \nabla h_{K^z}(\theta), \theta \rangle = h_{K^z}(\theta)$ and $\nabla h_{K^z}(\theta) \in K^z \subseteq L^z$, we must have

$$\nabla h_{K^z}(\theta) = \nabla h_{L^z}(\theta). \tag{20}$$

Hence, the theorem can be stated in the following form:

THEOREM 2.5. *Let K and L be smooth enough, and L be of maximal volume in K. For every $z \in \mathrm{int}(L)$, we can find contact points $v_1, \ldots, v_m$ of $K - z$ and $L - z$, contact points $u_1, \ldots, u_m$ of K^z and L^z, and positive reals $\lambda_1, \ldots, \lambda_m$, such that: $\sum \lambda_j u_j = o$, $\langle u_j, v_j \rangle = 1$, and*

$$\mathrm{Id} = \sum_{j=1}^{m} \lambda_j u_j \otimes v_j. \tag{21}$$

□

Remark. The analogue of the Dvoretzky–Rogers lemma [DR] in the context of Theorem 2.5 is the following: If F is a k-dimensional subspace of $\mathbb{R}^n$ and P_F denotes the orthogonal projection onto F, then there exists $j \in \{1, \ldots, m\}$ such that

$$\langle P_F(u_j), P_F(v_j) \rangle \geqslant \frac{k}{n}.$$

This can be easily checked, since

$$k = \mathrm{tr}P_F = \sum_{j=1}^{m} \lambda_j \langle P_F(u_j), P_F(v_j) \rangle,$$

and $\sum \lambda_j = n$.

As an application of Theorem 2.4, we give a direct proof of the fact that the diameter of the Banach–Mazur compactum is bounded by n:

PROPOSITION 2.6. *Let K and L be symmetric convex bodies in $\mathbb{R}^n$. Then, $d(K, L) \leqslant n$.*

Proof. We may assume that K and L satisfy our smoothness hypotheses, and that K is symmetric about o. Let L_1 be an affine image of L which is of maximal volume in K.

Claim. L_1 is also symmetric about o.

Let z be the center of L_1. Then $L_1 = 2z - L_1 \subseteq K$ and the symmetry of K shows that $L_1 - 2z \subseteq K$. It follows that

$$\tilde{L} = L_1 - z = \frac{L_1 + (L_1 - 2z)}{2} \subseteq K, \tag{22}$$

and $L_1 - z$ is o-symmetric. If $z \neq o$, we obtain a contradiction as follows: we define a linear map T which leaves $z^{\perp}$ unchanged and sends z to $(1+\alpha)z$, where $0 < \alpha < |z|^2/h_{L_1 - z}(z)$. One can easily check that

$$T(L_1 - z) \subseteq \mathrm{co}(L_1, L_1 - 2z) \subseteq K \quad \text{and} \quad |T(L_1 - z)| = (1+\alpha)|L_1| > |L_1|.$$

We write L for L_1. Let $x \in \mathbb{R}^n$ and choose $z = w = o$ and $T(y) = \langle \nabla h_{L^\circ}(x), y \rangle x$ in Lemma 2.3. Then there exists $\theta \in S^{n-1}$ such that $\rho_K(o, \theta) = \rho_L(o, \theta)$ and

$$\Big\langle \nabla h_{K^\circ}(\theta), \langle \nabla h_{L^\circ}(x), \rho_L(o, \theta)\theta \rangle x \Big\rangle \geqslant \frac{h_{L^\circ}(x)}{n}. \tag{23}$$

But, $\nabla h_{L^\circ}(x) \in L^\circ$ and $\rho_L(o, \theta)\theta \in L$. Since L is o-symmetric, we have

$$|\langle \nabla h_{L^\circ}(x), \rho_L(o, \theta)\theta \rangle| \leqslant 1. \tag{24}$$

Using now the o-symmetry of K and the fact that $\nabla h_{K^\circ}(\theta) \in K^\circ$, from (23) and (24) we get

$$h_{K^\circ}(x) \geqslant \frac{h_{L^\circ}(x)}{n}. \tag{25}$$

Therefore, $L^\circ \subseteq nK^\circ$, and this shows that $K \subseteq nL$. □

We now assume that L is symmetric and K is any convex body:

PROPOSITION 2.7. *Let L be a symmetric convex body and K be any convex body in $\mathbb{R}^n$. Then, $d(K, L) \leqslant 2n - 1$.*

Proof. We may assume that L is of maximal volume in K and L is symmetric about o.

Let $d > 0$ be the smallest positive real for which $h_{L^\circ}(y) \leqslant dh_{K^\circ}(y)$ for all $y \in \mathbb{R}^n$. Then, duality, the symmetry of L and the fact that $L \subseteq K$ show that $h_K(-x) \leqslant dh_L(-x) = dh_L(x) \leqslant dh_K(x)$ for every $x \in \mathbb{R}^n$.

We define $T(y) = \langle n\nabla h_{L^\circ}(x), y\rangle x$ and $w = \gamma x$, where $\gamma \in [0, n)$ is to be determined. From Lemma 2.3, there exists $\theta \in S^{n-1}$ such that $\rho_K(o, \theta) = \rho_L(o, \theta)$ and

$$\Big\langle \nabla h_{K^\circ}(\theta), \gamma x + n\langle \nabla h_{L^\circ}(x), \rho_L(o, \theta)\theta\rangle x\Big\rangle \geqslant \frac{n\langle \nabla h_{L^\circ}(x), x\rangle}{n} = h_{L^\circ}(x). \tag{26}$$

Since $\nabla h_{L^\circ}(x) \in L^\circ$, $\rho_L(o, \theta)\theta \in L$ and L is o-symmetric, we have

$$|\langle \nabla h_{L^\circ}(x), \rho_L(o, \theta)\theta\rangle| \leqslant 1,$$

therefore

$$\gamma - n \leqslant \gamma + n\langle \nabla h_{L^\circ}(x), \rho_L(o, \theta)\theta\rangle \leqslant \gamma + n. \tag{27}$$

Let $s = \langle \nabla h_{L^\circ}(x), \rho_L(o, \theta)\theta\rangle$. Since $\nabla h_{K^\circ}(x) \in K^\circ$, from (26) and (27) we get

$$h_{L^\circ}(x) \leqslant (\gamma + n)h_{K^\circ}(x), \tag{28}$$

if $\gamma + ns \geqslant 0$, and

$$h_{L^\circ}(x) \leqslant (n - \gamma)dh_{K^\circ}(x), \tag{29}$$

if $\gamma + ns < 0$. It follows that

$$h_{L^\circ}(x) \leqslant \max\{\gamma + n, (n - \gamma)d\}h_{K^\circ}(x). \tag{30}$$

This shows that $d \leqslant \max\{\gamma + n, (n - \gamma)d\}$, and choosing $\gamma = n(d-1)/(d+1)$ we get $d \leqslant 2n - 1$. Hence, $L^\circ \subseteq (2n-1)K^\circ$ and the result follows. $\square$

3. Choice of the Center

In this section we study the case where L is a polytope with vertices $v_1, \dots, v_N$, and K has C^2 boundary with strictly positive curvature ($K \in C^2_+$). Then, we can strengthen Theorem 2.5 in the following sense:

THEOREM 3.1. *Let L be of maximal volume in K. Then, there exists $z \in L\backslash\{v_1, \dots, v_N\}$ for which we can find $\lambda_1, \dots, \lambda_N \geqslant 0$, and $u_1, \dots, u_N \in \mathrm{bd}(K^z)$ so that*

(1) $\sum \lambda_j u_j = o, \quad \sum \frac{\lambda_j}{n} v_j = z.$
(2) $\langle u_j, v_j - z\rangle = 1$ for all $j = 1, \dots, N$.
(3) $\mathrm{Id} = \sum_{j=1}^N \lambda_j u_j \otimes v_j$.

Proof. We may assume that $o \in \operatorname{int}(L)$. By Theorem 2.4 and our hypotheses about K, for every $z \in L_0 := L \setminus \{v_1, \ldots, v_N\}$ there exist representations of the form

$$\mathrm{Id} = \sum_{j=1}^{N} \lambda_j u_j \otimes v_j,$$

where $\lambda_j \geqslant 0$, $u_j \in \mathrm{bd}(K^z)$ with $\langle u_j, v_j - z\rangle = 1$, and $\sum_{j=1}^{N} \lambda_j u_j = o$. Note that the representation of the identity follows from Theorem 2.4 because of the condition $\sum_{j=1}^{N} \lambda_j u_j = o$.

We define a set-function ϕ on L_0, setting $\phi(z)$ to be the set of all points $(1/n)\sum_{j=1}^{N} \lambda_j v_j \in L$ which come from such representations (with respect to z). The set $\phi(z)$ is clearly nonempty, convex and closed.

Let $s \in (0, 1)$. We define ϕ_s on L_0 with $\phi_s(z) = s\phi(z)$, and $g_s : L_0 \to \mathbb{R}^+$ with

$$g_s(z) = d(z, \phi_s(z)) = \inf\{|z - w| : \ w \in \phi_s(z)\}. \tag{1}$$

It is easily checked that ϕ_s is upper semi-continuous and g_s is lower semi-continuous.

LEMMA 3.2. *For every $s \in (0, 1)$, there exists $z \in sL$ such that $z \in \phi_s(z)$.*

Proof. Assume otherwise. Since $\phi_s(z) \subseteq sL$ for all $z \in L_0$, this means that $g_s(z) > 0$ on L_0. We set $r = (1 + s)/2$, and consider the restriction of ϕ_s onto rL. Since g_s is lower-semicontinuous, there exists $q = q(r, s) > 0$ such that $g_s(z) \geqslant q$ for all $z \in rL$.

On the other hand, ϕ_s is upper-semicontinuous, convex-valued with bounded range. Therefore, ϕ_s admits approximate continuous selections: By a result of Beer [Be] (see also [RW], pp. 195), for every $\varepsilon > 0$ there exists a continuous function $h_\varepsilon : rL \to \mathbb{R}^n$ so that

$$d(h_\varepsilon(z), s\phi(z)) < \varepsilon. \tag{2}$$

Let $c = c(r, s) > 0$ be such that $sL + cD_n \subseteq rL$. Letting $\varepsilon = (1/2)\min\{q, c\}$ we find continuous $h : rL \to rL$ satisfying (2). Brouwer's theorem shows that h has a fixed point $z \in rL$. But then,

$$q \leqslant d(z, s\phi(z)) = d(h(z), s\phi(z)) < \varepsilon,$$

which is a contradiction. This completes the proof. □

We apply Lemma 3.2 for a sequence $s_k \in (0, 1)$ with $s_k \to 1$. For each k we find $z_k \in s_k L$ and $\lambda_j^{(k)} \geqslant 0$ such that

$$\mathrm{Id} = \sum_{j=1}^{N} \lambda_j^{(k)} u_j^{(k)} \otimes v_j, \tag{3}$$

where $u_j^{(k)} \in \mathrm{bd}(K^{z_k})$ is uniquely determined by $\langle u_j^{(k)}, v_j - z_k\rangle = 1$, and

$$z_k = s_k \sum_{j=1}^{N} \frac{\lambda_j^{(k)}}{n} v_j, \qquad \sum_{j=1}^{N} \lambda_j^{(k)} u_j^{(k)} = o. \tag{4}$$

Passing to a subsequence, we may assume that $z_k \to z \in L$. If z is not one of the vertices of L, then $u_j^{(k)} \to u_j$, where $u_j \in \mathrm{bd}(K^z)$ and $\langle u_j, v_j - z\rangle = 1$. Passing to further subsequences we may assume that $\lambda_j^{(k)} \to \lambda_j \geqslant 0$. Since $s_k \to 1$, (3) and (4) imply

$$\mathrm{Id} = \sum_{j=1}^{N} \lambda_j u_j \otimes v_j, \tag{5}$$

and

$$z = \sum_{j=1}^{N} \frac{\lambda_j}{n} v_j, \qquad \sum_{j=1}^{N} \lambda_j u_j = o. \tag{6}$$

This is exactly the assertion of the Theorem, provided that we have proved the following:

CLAIM 3.3. *Let* $s_k \in (0, 1)$ *with* $s_k \to 1$, *and* $z_k \in s_k \phi(z_k)$. *If* $z_k \to z$, *then* $z \notin \{v_1, \ldots, v_N\}$.

Proof. We assume that z_k satisfy (3) and (4) and $z_k \to v_1$. Our assumptions about K imply that K^{v_1} is unbounded only in the direction of $N(v_1)$, where $N(v_1)$ is the unit normal vector to K at v_1. For large k, z_k is away from $v_2, \ldots, v_N$, therefore $u_j^{(k)} \to u_j$, $j = 2, \ldots, N$, where u_j is the unique point in $\mathrm{bd}(K^{v_1})$ for which $\langle u_j, v_j - z\rangle = 1$.

Since $(u_j^{(k)}), j \geqslant 2$ is bounded and $\sum_{j=1}^{N} \lambda_j = n$, (4) shows that

$$|\lambda_1^{(k)} u_1^{(k)}| = |\sum_{j=2}^{N} \lambda_j^{(k)} u_j^{(k)}|$$

remains bounded. Hence, passing to a subsequence we may assume that $\lambda_1^{(k)} u_1^{(k)} \to w_1$, and $\lambda_j^{(k)} \to \lambda_j$ for all $j = 1, \ldots, N$. This means that

$$\mathrm{Id} = w_1 \otimes v_1 + \sum_{j=2}^{N} \lambda_j u_j \otimes v_j, \tag{7}$$

and

$$v_1 = \sum_{j=1}^{N} \frac{\lambda_j}{n} v_j, \qquad w_1 + \sum_{j=2}^{N} \lambda_j u_j = o. \tag{8}$$

Since v_1 is a vertex of L, we must have $\lambda_2 = \ldots = \lambda_N = 0$. Then, $w_1 = o$, and (7) takes the form $\mathrm{Id} = 0$, which is a contradiction. □

Actually, the argument we used for the proof of Claim 3.3 shows the following extension of Theorem 2.4:

PROPOSITION 3.4. *Let K, L be smooth enough and assume that L is of maximal volume inside K. For every $z \in \mathrm{bd}(K) \cap \mathrm{bd}(L)$, there exist $m_0 \leqslant m \leqslant n^2 + n + 1$, contact points $v_1, \ldots, v_m$ of K and L, contact points $u_1, \ldots, u_{m_0}$ of K^z and L^z, and nonnegative numbers $\lambda_1, \ldots, \lambda_{m_0}$, $\alpha_{m_0+1}, \ldots, \alpha_m$ so that:*

(1) $\langle u_j, v_j - z\rangle = 1$ for all $j = 1, 2, \ldots, m_0$,

(2) $\langle \alpha_j N(z), v_j - z\rangle = 0$ for all $j = m_0 + 1, \ldots, m$,

(3) $\mathrm{Id} = \sum_{j=1}^{m_0} \lambda_j u_j \otimes v_j + N(z) \otimes (\sum_{j=m_0+1}^{m} \alpha_j v_j)$,

where $N(z)$ is the unit normal vector of K at z.

Sketch of the proof. Let $z \in \mathrm{bd}(K) \cap \mathrm{bd}(L)$, and consider a sequence $z_k \in \mathrm{int}(L)$ with $z_k \to z$. Applying Theorem 2.4, for each k we find $\lambda_j^{(k)} \geqslant 0$, contact points $v_j^{(k)}$ of K and L, and contact points $u_j^{(k)}$ of K^{z_k} and L^{z_k} so that

$$\sum_{j=1}^{N} \lambda_j^{(k)} u_j^{(k)} = o, \left\langle u_j^{(k)}, v_j^{(k)} - z_k \right\rangle = 1 \quad \text{and} \quad \mathrm{Id} = \sum_{j=1}^{N} \lambda_j^{(k)} u_j^{(k)} \otimes v_j^{(k)}.$$

We may assume that $N = n^2 + n + 1$ for all k.

Passing to subsequences we may assume that $\lambda_j^{(k)} \to \lambda_j$ and $v_j^{(k)} \to v_j$ as $k \to \infty$, where $\lambda_j \geqslant 0$ and v_j are contact points of K and L. We may also assume that there exists $m_0 \leqslant N$ such that $u_j^{(k)} \to u_j$ if $j \leqslant m_0$, and $|u_j^{(k)}| \to \infty$ if $j > m_0$.

Let $N(z)$ be the unit normal vector to K at z. It is not hard to see that for all $j > m_0$, the angle between $u_j^{(k)}$ and $N(z)$ tends to zero as $k \to \infty$. Using the fact that $\sum_{j=1}^{N} \lambda_j^{(k)} u_j^{(k)} = o$, we then see that for large k

$$\max_{j > m_0} |\lambda_j^{(k)} u_j^{(k)}| \leqslant |\sum_{j > m_0} \lambda_j^{(k)} u_j^{(k)}| = |\sum_{j \leqslant m_0} \lambda_j^{(k)} u_j^{(k)}|, \tag{9}$$

and this quantity remains bounded, since all $\lambda_j^{(k)}$ and $u_j^{(k)}$ $(j \leqslant m_0)$ converge. Therefore, we may also assume that $\lambda_j^{(k)} u_j^{(k)} \to \alpha_j N(z)$, $j > m_0$.

Passing to the limit we check that $\langle u_j, v_j - z\rangle = 1$, $j \leqslant m_0$, and

$$\mathrm{Id} = \sum_{j=1}^{m_0} \lambda_j u_j \otimes v_j + N(z) \otimes \left(\sum_{j=m_0+1}^{N} \alpha_j v_j \right). \tag{10}$$

Finally,

$$\langle \alpha_j N(z), v_j - z\rangle = \lim_k \lambda_j^{(k)} \left\langle u_j^{(k)}, v_j^{(k)} - z_k \right\rangle = \lim_k \lambda_j^{(k)} = 0$$

for all $j > m_0$, and

$$\sum_{j=1}^{m_0} \lambda_j u_j + \left(\sum_{j=m_0+1}^{N} \alpha_j \right) N(z) = o.$$

Ignoring all j's for which $\alpha_j = 0$, we conclude the proof. □

4. Volume Ratio

In this Section we give an estimate for the volume ratio of two convex bodies:

THEOREM 4.1. *Let L be of maximal volume in K. Then,* $(|K|/|L|)^{1/n} \leqslant n$.

Proof. Without loss of generality we may assume L is a polytope and $K \in C_+^2$, and using Theorem 3.1 we may assume that $o \in L \cap \mathrm{int}(K)$, and

$$\mathrm{Id} = \sum_{j=1}^{m} \lambda_j u_j \otimes v_j, \tag{1}$$

where $\lambda_j > 0$, $u_1, \ldots, u_m \in \mathrm{bd}(K^\circ)$, $v_1, \ldots, v_m$ are contact points of K and L, $\langle u_j, v_j \rangle = 1$, and $\sum_{j=1}^{m} \lambda_j u_j = o = \sum_{j=1}^{m} \lambda_j v_j$. This last condition shows that $m \geqslant n+1$.

Since $u_j \in K^\circ$, $j = 1, \ldots, m$, we have the inclusion

$$K \subseteq U := \{x : \langle x, u_j \rangle \leqslant 1,\ j = 1, \ldots, m\}. \tag{2}$$

Observe that U is a convex body, because $\sum \lambda_j u_j = o$. On the other hand, $v_j \in L$, $j = 1, \ldots, m$. Therefore,

$$L \supseteq V := \mathrm{co}\{v_1, \ldots, v_m\}. \tag{3}$$

It follows that

$$\frac{|K|}{|L|} \leqslant \frac{|U|}{|V|}. \tag{4}$$

We define $\tilde{v}_j \in \mathbb{R}^{n+1}$ by

$$\tilde{v}_j = \frac{n}{n+1}(-v_j, 1), \quad j = 1, \ldots, m. \tag{5}$$

Then, we can estimate $|V|$ using the reverse form of the Brascamp–Lieb inequality (see [Bar]):

LEMMA 4.2. *Let*

$$D_{\tilde{v}} = \inf\left\{ \frac{\det\left(\sum_{j=1}^{m} \lambda_j \alpha_j v_j \otimes v_j\right)}{\prod_{j=1}^{m} \alpha_j^{\lambda_j}} : \alpha_j > 0,\ j = 1, 2, \ldots, m \right\}.$$

Then, the volume of V satisfies the inequality

$$|V| \geqslant \left(\frac{n+1}{n}\right)^{n+1} \frac{\sqrt{D_{\tilde{v}}}}{n!}. \tag{6}$$

Proof. Let

$$N_V(x) = \begin{cases} \inf\left\{\sum_{i=1}^m \alpha_i : \alpha_i \geqslant 0 \text{ and } x = \sum_{i=1}^m \alpha_i \tilde{v}_i\right\}, & \text{if such } \alpha_i \text{ exist,} \\ +\infty, & \text{otherwise.} \end{cases}$$

Let also $C = \mathrm{co}\{-v_1, -v_2, \dots, -v_m\}$.

Claim. If $x = (y, r)$ for some $y \in \mathbb{R}^n$ and $r \in \mathbb{R}$, then

$$\mathrm{e}^{-N_V(x)} \leqslant \chi_{\{y \in rC\}} \chi_{\{r \geqslant 0\}} \mathrm{e}^{-\frac{n+1}{n} r}. \tag{7}$$

[If $r < 0$ then $N_V(x) = +\infty$ and the inequality is true. Otherwise, let $\alpha_i \geqslant 0$ be such that $x = \sum_{i=1}^m \alpha_i \tilde{v}_i$ and $\sum_{i=1}^m \alpha_i = N_V(x)$. Then, it is immediate that $N_V(x) = \big((n+1)/n\big) r \geqslant 0$ and $y = \big(n/(n+1)\big) \sum_{i=1}^m \alpha_i(-v_i) \in rC$. From this (7) follows.]

Integrating the inequality (7) we get

$$\int_{\mathbb{R}^{n+1}} \mathrm{e}^{-N_V(x)}\, \mathrm{d}x \leqslant n! \left(\frac{n}{n+1}\right)^{n+1} |V|.$$

We now set $d_j = \big((n+1)/n\big)\lambda_j$ and apply the reverse form of the Brascamp–Lieb inequality to the left hand side integral:

$$\begin{aligned} \int_{\mathbb{R}^{n+1}} \mathrm{e}^{-N_V(x)}\, \mathrm{d}x &= \int_{\mathbb{R}^{n+1}} \sup_{\substack{\alpha_j \geqslant 0 \\ x = \sum_{j=1}^m \alpha_j \tilde{v}_j}} \prod_{j=1}^m \mathrm{e}^{-\alpha_j}\, \mathrm{d}x \\ &= \int_{\mathbb{R}^{n+1}} \sup_{x = \sum_{j=1}^m \alpha_j \tilde{v}_j} \prod_{j=1}^m \left(\mathrm{e}^{-\alpha_j/d_j} \chi_{\{\alpha_j \geqslant 0\}}\right)^{d_j} \\ &\geqslant \sqrt{D_{\tilde{v}}} \prod_{j=1}^m \left(\int_0^\infty \mathrm{e}^{-t}\, \mathrm{d}t\right)^{d_j} = \sqrt{D_{\tilde{v}}}. \end{aligned}$$

From this (6) follows. □

We now turn to find an upper bound for $|U|$: as above, let $d_j = \big((n+1)/n\big)\lambda_j$ and set $\tilde{u}_j = \big(-u_j, (1/n)\big)$ for $j = 1, \dots, m$.

LEMMA 4.3. *The volume of U satisfies the inequality*

$$|U| \leqslant \frac{1}{\sqrt{D_{\tilde{u}}}} \frac{(n+1)^{n+1}}{n!n}, \tag{8}$$

where

$$D_{\tilde{u}} = \inf\left\{ \frac{\det(\sum d_j \alpha_j \tilde{u}_j \otimes \tilde{u}_j)}{\prod \alpha_j^{d_j}}; \alpha_j > 0 \right\}. \tag{9}$$

Proof. We apply the Brascamp–Lieb inequality [BL] (see also [Bar]) in the spirit of K. Ball's proof of the fact that among all convex bodies having the Euclidean unit ball as their ellipsoid of maximal volume, the regular simplex has maximal volume [Ba].

For each $j = 1, \ldots, m$, define $f_j \colon \mathbb{R} \to [0, \infty)$ by $f_j(t) = \mathrm{e}^{-t}\chi_{[0,\infty)}(t)$, and set

$$F(x) = \prod_{j=1}^{m} f_j(\langle \tilde{u}_j, x \rangle)^{d_j}, \quad x \in \mathbb{R}^{n+1}. \tag{10}$$

The Brascamp–Lieb inequality gives

$$\int_{\mathbb{R}^{n+1}} F(x)\mathrm{d}x \leqslant \frac{1}{\sqrt{D_{\tilde{u}}}} \prod_{j=1}^{m} \left(\int_{\mathbb{R}} f_j \right)^{d_j} = \frac{1}{\sqrt{D_{\tilde{u}}}}. \tag{11}$$

As in [Ba], writing $x = (y, r) \in \mathbb{R}^n \times \mathbb{R}$, we see that $F(x) = 0$ if $r < 0$. When $r \geqslant 0$, we have $F(x) \neq 0$ precisely when $y \in (r/n)U$, and then, taking into account the facts that $\sum \lambda_j u_j = o$ and $\sum d_j = n + 1$, we see that F is independent of y and equal to

$$F(x) = \exp(-r(n+1)/n). \tag{12}$$

It follows from (11) that

$$\frac{1}{\sqrt{D_{\tilde{u}}}} \geqslant \int_0^{\infty} \exp(-r(n+1)/n) \left(\frac{r}{n}\right)^n |U| \mathrm{d}r = |U| \frac{n!n}{(n+1)^{n+1}}. \qquad \square \tag{13}$$

Combining the two lemmata, we get

$$\frac{|K|}{|L|} \leqslant \frac{n^n}{\sqrt{D_{\tilde{u}} D_{\tilde{v}}}}. \tag{14}$$

Observe that $\tilde{u}_j$, $\tilde{v}_j$ and d_j satisfy $\langle \tilde{u}_j, \tilde{v}_j \rangle = 1$, $j = 1, \ldots, m$. Using the fact that $\sum_{j=1}^{m} \lambda_j u_j = o = \sum_{j=1}^{m} \lambda_j v_j$, we check that

$$\mathrm{Id} = \sum_{j=1}^{m} d_j \tilde{u}_j \otimes \tilde{v}_j.$$

Thus, in order to finish the proof of Theorem 4.1 it suffices to prove the following proposition.

PROPOSITION 4.4. *Let* $\lambda_1, \dots, \lambda_m > 0$, $u_1, \dots u_m$ *and* $v_1, \dots, v_m$ *be vectors satisfying* $\langle u_j, v_j \rangle = 1$ *for all* $j = 1, \dots m$ *and*

$$\mathrm{Id} = \sum_{j=1}^{m} \lambda_j u_j \otimes v_j. \tag{15}$$

Then $D_u D_v \geqslant 1$.

Proof. For $I \subseteq \{1, 2, \dots, m\}$ we use the notation $\lambda_I = \prod_{i \in I} \lambda_i$, $\alpha_I = \prod_{i \in I} \alpha_i$, and for I's with cardinality n, we write $U_I = \det(u_i : i \in I)$ and $V_I = \det(v_i : i \in I)$. Moreover, we write $(\lambda U)_I$ for $\det(\lambda_i u_i : i \in I)$.

Applying the Cauchy–Binet formula we have

$$\det\left(\sum_{j=1}^{m} \lambda_j \alpha_j u_j \otimes v_j\right) = \sum_{\substack{|I|=n \\ I \subseteq \{1,2,\dots,m\}}} \alpha_I (\sqrt{\lambda} U)_I (\sqrt{\lambda} V)_I. \tag{16}$$

But

$$\sum (\sqrt{\lambda} U)_I (\sqrt{\lambda} V)_I = \det\left(\sum_{j=1}^{m} \lambda_j u_j \otimes v_j\right) = \det(\mathrm{Id}) = 1.$$

Hence, applying the arithmetic-geometric means inequality to the right side of (16) we deduce that

$$\begin{aligned} \sum_{\substack{|I|=n \\ I \subseteq \{1,2,\dots,m\}}} \alpha_I (\sqrt{\lambda} U)_I (\sqrt{\lambda} V)_I &\geqslant \prod_{\substack{|I|=n \\ I \subseteq \{1,2,\dots,m\}}} \alpha_I^{(\sqrt{\lambda} U)_I (\sqrt{\lambda} V)_I} \\ &= \prod_{j=1}^{m} \alpha_j^{\sum_{j \in I, |I|=n} (\sqrt{\lambda} U)_I (\sqrt{\lambda} V)_I}. \end{aligned}$$

Observe now that the exponent of α_j in the above product equals λ_j:

$$\begin{aligned} \sum_{j \in I,\ |I|=n} (\sqrt{\lambda} U)_I (\sqrt{\lambda} V)_I &= \sum_{|I|=n} (\sqrt{\lambda} U)_I (\sqrt{\lambda} V)_I - \sum_{j \notin I,\ |I|=n} (\sqrt{\lambda} U)_I (\sqrt{\lambda} V)_I \\ &= \det\left(\sum_{j=1}^{m} \lambda_j u_j \otimes v_j\right) - \det(I - \lambda_j u_j \otimes v_j) \\ &= \lambda_j, \end{aligned}$$

since $\langle u_j, v_j \rangle = 1$. Thus, we have shown that

$$\det\left(\sum_{j=1}^{m} \lambda_j \alpha_j u_j \otimes v_j\right) \geqslant \prod_{j=1}^{m} \alpha_j^{\lambda_j}. \tag{17}$$

Now, for any $\gamma_j,\ \delta_j > 0$ we have

$$\det\left(\sum_{j=1}^{m}\lambda_j\gamma_j u_j\otimes u_j\right)\det\left(\sum_{j=1}^{m}\lambda_j\delta_j v_j\otimes v_j\right) = \sum_{|I|=n}\gamma_I(\sqrt{\lambda}U)_I^2\sum_{|I|=n}\delta_I(\sqrt{\lambda}V)_I^2.$$

By the Cauchy–Schwarz inequality this is greater than

$$\left(\sum_{|I|=n}\lambda_I\sqrt{\gamma_I\delta_I}\,U_I V_I\right)^2.$$

Apply now (17) to get

$$\frac{\det\left(\sum_{j=1}^{m}\lambda_j\gamma_j u_j\otimes u_j\right)}{\prod_{j=1}^{m}\gamma_j^{\lambda_j}}\frac{\det\left(\sum_{j=1}^{m}\lambda_j\delta_j v_j\otimes v_j\right)}{\prod_{j=1}^{m}\delta_j^{\lambda_j}} \geqslant 1,$$

completing the proof. □

Remark. A different argument shows that $\mathrm{vr}(K, S_n) \leqslant c\sqrt{n}$ for every convex body K in $\mathbb{R}^n$, where $c > 0$ is an absolute constant.

Without loss of generality, we may assume that K is of maximal volume in D_n. Then, John's theorem gives us $\lambda_1, \ldots, \lambda_m > 0$ and contact points $u_1, \ldots, u_m$ of K and D_n such that

$$\mathrm{Id} = \sum_{j=1}^{m}\lambda_j u_j\otimes u_j.$$

The Dvoretzky–Rogers lemma [DR] shows that we can choose $u_1, \ldots, u_n$ among the u_j's so that

$$|P_{\mathrm{span}\{u_s:s<i\}^{\perp}}u_i| \geqslant \left(\frac{n-i+1}{n}\right)^{1/2}, \quad i = 2, \ldots, n.$$

Therefore, the simplex $S = \mathrm{co}\{o, u_1, \ldots, u_n\}$ has volume

$$|S| \geqslant \frac{1}{n!}\prod_{i=2}^{n}\left(\frac{n-i+1}{n}\right)^{1/2} = \frac{1}{(n!n^n)^{1/2}},$$

and $S \subseteq K \subseteq D_n$. It follows that

$$\mathrm{vr}(K, S_n) \leqslant \left(\frac{|D_n|}{|S|}\right)^{1/n} \leqslant \frac{(n!)^{1/2n}\sqrt{n}\sqrt{\pi}}{[\Gamma(\frac{n}{2}+1)]^{1/n}} \leqslant c\sqrt{n}.$$

This supports the question if $\mathrm{vr}(K, L)$ is always bounded by $c\sqrt{n}$.

Acknowledgements

This work was partially supported by a research grant of the University of Crete. The first named author acknowledges the hospitality of the Erwin Schrödinger International Institute for Mathematical Physics in Vienna. The second named author was supported by the Hellenic State Scholarship Foundation. Finally, we thank J. Bastéro and M. Romance for pointing out an error in an earlier version of this paper: their remark pushed us to prove the results in Section 3.

References

[Ba] Ball, K. M.: Volume ratios and a reverse isoperimetric inequality, *J. London Math. Soc. (2)* **44** (1991), 351–359.

[Bar] Barthe, F.: Inégalités de Brascamp–Lieb et convexité, *C.R. Acad. Sci. Paris* **324** (1997), 885–888.

[Be] Beer, G.: Approximate selections for upper semicontinuous convex valued multifunctions, *J. Approx. Theory* **39** (1983), 172–184.

[BL] Brascamp, H. and Lieb, E. H.: Best constants in Young's inequality, its converse and its generalization to more than three functions, *Adv. in Math.* **20** (1976), 151–173.

[DR] Dvoretzky, A. and Rogers, C. A.: Absolute and unconditional convergence in normed linear spaces, *Proc. Nat. Acad. Sci. USA* **36** (1950), 192–197.

[Gl] Gluskin, E. D.: The diameter of the Minkowski compactum is approximately equal to *n*, *Funct. Anal. Appl.* **15** (1981), 72–73.

[Gr] Grünbaum, B.: Measures of symmetry for convex sets, In: *Convexity VII*, Proc. Sympos. Pure Math., Amer. Math. Soc., Providence, 1963, pp. 233–270.

[GM] Giannopoulos, A. A. and Milman, V. D.: Extremal problems and isotropic positions of convex bodies, *Israel J. Math.* **117** (2000), 29–60.

[J] John, F.: *Extremum Problems with Inequalities as Subsidiary Conditions*, Courant Anniversary Volume, New York, 1948, pp. 187–204.

[L] Lassak, M.: Approximation of convex bodies by centrally symmetric bodies, *Geom. Dedicata* **72** (1998), 63–68.

[LTJ] Litvak, A. and Tomczak-Jaegermann, N.: Random aspects of high-dimensional convex bodies, In: Milman and Schechtman (eds), *Lecture Notes in Math*. 1745, Springer, New York, 2000, pp. 168–190.

[P] Palmon, O.: The only convex body with extremal distance from the ball is the simplex, *Israel J. Math.* **80** (1992), 337–349.

[R] Rudelson, M.: Distances between non-symmetric convex bodies and the MM*-estimate, *Positivity* **4** (2000), 161–178.

[RW] Rockafellar, R. T. and Wets, R. J-B.: *Variational Analysis*, Grundl. Math. Wissen. **317**, Springer, Berlin, 1998.

[TJ] Tomczak-Jaegermann, N.: *Banach–Mazur Distances and Finite Dimensional Operator Ideals*, Pitman Monographs 38, Pitman, London, 1989.